Section 2.2
Limits
2 Lectures

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MATHS 101: Calculus I

Definition of a limit

Example

How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave at $x = 1$?

Solution:

$$f(1) = \frac{1^2 - 1}{1 - 1} = \frac{0}{0} \quad \text{undefined!}$$

Hence, we cannot substitute directly with $x = 1$, so instead we check values that are *very much close* to $x = 1$ and we check the corresponding values of $f(x)$.

x	0.9	0.99	0.9999	1	1.00001	1.001	1.01
$f(x)$				-			

Continue...

So we have seen that as x approaches 1, $f(x)$ approaches 2, we write

$$\lim_{x \to 1} f(x) = 2$$

Exercise

Using the method of the previous example (the table) find the following limits:

1. $\lim_{x\to 5} x$.
2. $\lim_{x\to a} x$.
3. $\lim_{x\to a} 7$.
4. $\lim_{x\to a} k$.
5. $\lim_{x\to 0} f(x)$, where

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Question: In order to find the limit of a function, do we need to do the table method every time?

To find the limit $\lim_{x \to a} f(x)$, we have

1. Substitute directly by $x = a$ in $f(x)$. If you get a real number, then that is the limit.
2. If you get undefined values such as $\frac{0}{0}$, we use *algebraic method* to clear any problem.

Zero denominator of a rational function

A - Eliminating zero denominator by canceling common factor in the numerator and denominator ($\frac{0}{0}$ form).

Example

Find

$$\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$$

Solution: Direct substitution gives

$$\frac{5^2 - 25}{5 - 5} = \frac{0}{0} \quad \text{undefined!}$$

So we factor both the denominator and numerator to cancel the common zero.

$$\begin{aligned}\lim_{x \to 5} \frac{x^2 - 5}{x - 5} &= \lim_{x \to 5} \frac{(x-5)(x+5)}{(x-5)} \\ &= \lim_{x \to 5} (x+5) \\ &= 5 + 5 = 10\end{aligned}$$

Exercise

Find

$$\lim_{x \to 1} \frac{x^2 - 2x + 1}{x - 1}$$

Example

Find

$$\lim_{x\to 3} \frac{x^2-3x}{x^2-9}$$

Solution: Direct substitution gives

$$\frac{0^2-3(0)}{0^2-9} = \frac{0}{0} \quad \text{undefined!}$$

So we factor both the denominator and numerator to cancel the common zero.

$$\begin{aligned}\lim_{x\to 3} \frac{x^2-3x}{x^2-9} &= \lim_{x\to 3} \frac{x(x-3)}{(x-3)(x+3)} \\ &= \lim_{x\to 3} \frac{x}{x+3} \\ &= \frac{3}{6} = \frac{1}{2}\end{aligned}$$

Example

Find

$$\lim_{y\to 0} \frac{5y^3+8y^2}{3y^4-16y^2}$$

Solution: Direct substitution gives

$$\frac{5(0)^3+8(0)^2}{3(0)^4-16(0)^2} = \frac{0}{0} \quad \text{undefined!}$$

So we factor both the denominator and numerator to cancel the common zero.

$$\begin{aligned}\lim_{y\to 0} \frac{5y^3+8y^2}{3y^4-16y^2} &= \lim_{y\to 0} \frac{y^2(5y+8)}{y^2(3y^2-16)} \\ &= \lim_{y\to 0} \frac{5y+8}{3y^2-16} \\ &= \frac{5(0)+8}{3(0)^2-16} = \frac{8}{-16} = \frac{-1}{2}\end{aligned}$$

Exercise

Find

$$\lim_{x \to 1} \frac{x^2 - 3x + 4}{x^2 - 6x + 7}$$

Example

Find

$$\lim_{x\to -1} \frac{2x^2+3x+1}{x^2-2x-3}$$

Solution: Direct substitution gives

$$\frac{2(0)^2+3(0)+1}{(0)^2-2(0)-3} = \frac{0}{0} \quad \text{undefined!}$$

So we factor both the denominator and numerator to cancel the common zero.

$$\begin{aligned}\lim_{x\to -1} \frac{2x^2+3x+1}{x^2-2x-3} &= \lim_{x\to -1} \frac{2(x+\frac{1}{2})(x+1)}{(x-3)(x+1)} \\ &= \lim_{x\to -1} \frac{2(x+\frac{1}{2})}{x-3} \\ &= \frac{2(-\frac{1}{2})}{-4} = \frac{1}{4}\end{aligned}$$

Exercise

Find

$$\lim_{x \to 2} \frac{x^3 - 8}{3x^2 - x - 10}$$

Exercise

Find

$$\lim_{x\to 1} \frac{x^2 - 3x + 4}{x^2 - 6x + 7}$$

Example

Find

$$\lim_{x\to 1} \frac{4x^5-4}{5x^2-5}$$

Solution: Direct substitution gives

$$\frac{4(0)^5-4}{5(0)^2-5} = \frac{0}{0} \quad \text{undefined!}$$

So we factor both the denominator and numerator to cancel the common zero.

$$\begin{aligned}
\lim_{x\to 1} \frac{4x^5-4}{5x^2-5} &= \lim_{x\to 1} \frac{4(x^5-1)}{5(x^2-1)} \\
&= \lim_{x\to 1} \frac{4(x-1)(x^4+x^3+x^2+x+1)}{5(x-1)(x+1)} \\
&= \lim_{x\to 1} \frac{4(x^4+x^3+x^2+x+1)}{5(x+1)} \\
&= \frac{20}{10} = 2
\end{aligned}$$

Example

Find

$$\lim_{x \to 1} \frac{\sqrt{2}}{3}$$

Solution: Direct substitution gives

$$\frac{\sqrt{2}}{3}$$

Exercise

Find

$$\lim_{x\to 2} \frac{x^4 - 16}{x^3 - 8}$$

Exercise

Find

$$\lim_{x \to 5} \frac{5x - 25}{x^2 - 10 + 25}$$

Conjugate and multiply by 1

B - Eliminating zero denominator by multiplying with the conjugate.

Example

Find

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$$

Solution: Direct substitution gives

$$\frac{\sqrt{9} - 3}{9 - 9} = \frac{0}{0} \quad \text{undefined!}$$

Since there is a square root and we cannot factor anything, we multiply both the numerator and denominator with the conjugate of the numerator (the one that contains the square root).

Continue...

$$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9}\frac{(\sqrt{x}-3)}{(x-9)}\frac{(\sqrt{x}+3)}{(\sqrt{x}+3)}$$
$$= \lim_{x\to 9}\frac{x-9}{(x-9)(\sqrt{x}+3)}$$
$$= \lim_{x\to 9}\frac{1}{(\sqrt{x}+3)}$$
$$= \frac{1}{6}$$

Exercise

Find

$$\lim_{x \to 9} \frac{\sqrt{x-5}-2}{x-9}$$

Conjugate and multiply by 1

Example

Find

$$\lim_{x \to 7} \frac{\sqrt{x+2} - 3}{x^2 - 49}$$

Solution: Direct substitution gives

$$\frac{\sqrt{9} - 3}{49 - 49} = \frac{0}{0} \quad \text{undefined!}$$

Since there is a square root and we cannot factor anything, we multiply both the numerator and denominator with the conjugate of the numerator (the one that contains the square root).

Continue...

$$\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x^2-49} = \lim_{x\to 7} \frac{(\sqrt{x+2}-3)}{(x^2-49)} \frac{(\sqrt{x+2}+3)}{(\sqrt{x+2}+3)}$$

$$= \lim_{x\to 7} \frac{x+2-9}{(x-7)(x+7)(\sqrt{x+2}+3)}$$

$$= \lim_{x\to 7} \frac{(x-7)}{(x-7)(x+7)(\sqrt{x+2}+3)}$$

$$= \lim_{x\to 7} \frac{1}{(x+7)(\sqrt{x+2}+3)}$$

$$= \frac{1}{84}$$

Exercise

Find

$$\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2}$$

Properties of Limits

Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$.

1. $\lim_{x\to a} c = c$.
2. $\lim_{x\to a} x^n = a^n$.
3. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + K$.
4. $\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot K$.
5. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x) = cL$.
6. $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{K}$ if $K \neq 0$.
7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$. If n is even, then L must be non–negative.

Summary: These properties are telling us we can substitute directly with the value of a if there no problem.

Exercise

Find

$$\lim_{x \to -3} \left(\frac{x+3}{x^2-9} \right)^{102}$$

The Sandwich Theorem (Squeeze theorem)

Theorem

Suppose $g(x) \leq f(x) \leq h(x)$, *i.e.,* f *is squeezed between* g *and* h. *Suppose also that*

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

Then we must have

$$\lim_{x \to a} f(x) = L$$

Example

Find $\lim_{x\to 0} u(x)$ if $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$.

Solution:

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$$

$$\lim_{x\to 0} 1 - \frac{x^2}{4} \leq \lim_{x\to 0} u(x) \leq \lim_{x\to 0} 1 + \frac{x^2}{2}$$

$$1 \leq \lim_{x\to 0} u(x) \leq 1$$

$$\lim_{x\to 0} u(x) = 1$$

Example

Find $\lim_{x \to 0} f(x)$ if $\sqrt{5-2x^2} \leq f(x) \leq \sqrt{5-x^2}$.

Solution:

$$\sqrt{5-2x^2} \leq f(x) \leq \sqrt{5-x^2}$$

$$\lim_{x \to 0} \sqrt{5-2x^2} \leq \lim_{x \to 0} f(x) \leq \lim_{x \to 0} \sqrt{5-x^2}$$

$$\sqrt{5} \leq \lim_{x \to 0} f(x) \leq \sqrt{5}$$

$$\lim_{x \to 0} f(x) = \sqrt{5}$$