Section 3.3
Basic Derivatives
1 Lecture

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MATHS 101: Calculus I

1. Differentiation formula for the basic functions.
2. Differentiation Rules.
   - Sum and constant multiple rule.
   - Product and Quotient Rules.

Definition of the derivative

Recall: As in the homework, we find that

1. $\frac{d}{dx}\left(c\right) = 0.$
2. $\frac{d}{dx}\left(x\right) = 1.$
3. $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}.$
4. $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}.$

Next we want to find the derivative of the power function $f(x) = x^n$, for any non-negative integer.

We will use the following:

- $z^n - x^n =$
  $(z-x)(\underbrace{z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \cdots + z^2x^{n-3} + zx^{n-2} + x^{n-1}}_{n \text{ terms each of power } n-1}).$
- $f'(x) = \lim_{z \to x} \frac{f(z)-f(x)}{z-x}$

Power Rule

Theorem

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Let $f(x) = x^n$, then

$$
\begin{aligned}
f'(x) &= \lim_{z\to x} \frac{f(z) - f(x)}{z - x} \\
&= \lim_{z\to x} \frac{z^n - x^n}{z - x} \\
&= \lim_{z\to x} \frac{(z-x)\underbrace{(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})}_{n \text{ terms each of power } n-1}}{(z-x)} \\
&= \lim_{z\to x} \underbrace{(z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \cdots + z^2x^{n-3} + zx^{n-2} + x^{n-1})}_{n \text{ terms each of power } n-1} \\
&= \underbrace{x^{n-1} + x^{n-2}x + x^{n-3}x^2 + \cdots + x^2x^{n-3} + xx^{n-2} + x^{n-1})}_{n \text{ terms of power } n-1}
\end{aligned}
$$

Continue...

$$= \lim_{z \to x} \underbrace{\left(z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \cdots + z^2x^{n-3} + zx^{n-2} + x^{n-1} \right)}_{n \text{ terms each of power } n-1}$$

$$= \underbrace{x^{n-1} + x^{n-2}x + x^{n-3}x^2 + \cdots + x^2x^{n-3} + xx^{n-2} + x^{n-1})}_{n \text{ terms of power } n-1}$$

$$= \underbrace{x^{n-1} + x^{n-1} + x^{n-1} + \cdots + x^{n-1} + x^{n-1} + x^{n-1})}_{n \text{ terms of power } n-1}$$

$$= nx^{n-1}$$

Example

1. $\frac{d}{dx}\left(x^5\right) = 5x^4$.
2. $\frac{d}{dx}\left(x^2\right) = 2x$.
3. $\frac{d}{dx}\left(x^3\right) = 3x^2$.
4. $\frac{d}{dx}\left(x\right) = 1x^0 = 1$.
5. $\frac{d}{dx}\left(x^\pi\right) = \pi x^{\pi-1}$.
6. $\frac{d}{dx}\left(x^{-10}\right) = -10x^{-10-1} = -10x^{-11}$.
7. $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{-\frac{1}{2}}\right) = -\frac{1}{2}x^{-\frac{1}{2}}$.
8. $\frac{d}{dx}\left(\sqrt[6]{x^7}\right) = \frac{d}{dx}\left(x^{\frac{7}{6}}\right) = -\frac{7}{6}x^{\frac{1}{6}}$.
9. $\frac{d}{dx}\left(\frac{1}{x^3\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{1}{x^{\frac{7}{2}}}\right) = \frac{d}{dx}\left(x^{-\frac{7}{2}}\right) = -\frac{7}{2}x^{-\frac{7}{2}-1} = -\frac{7}{2}x^{-\frac{-9}{2}}$.

Exponential Functions

Theorem

Let $f(x) = a^x$*, then*

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Proof:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h \to 0} a^x \cdot \frac{a^h - 1}{h} \\ &= a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h} \end{aligned}$$

Continue...

$$= a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}$$
$$= a^x \cdot \ln a$$

Note: Let $a = e = 2.71828281\ldots$ be the Euler number, then

$$\frac{d}{dx}(e^x) = e^x \ln e = e^x$$

Theorem

Let $f(x) = e^{kx}$, then

$$\frac{d}{dx}(e^{kx}) = ke^{kx}$$

Proof:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{e^{kx+kh} - e^{kx}}{h} \\
&= \lim_{h\to 0} \frac{e^{kx}a^{kh} - e^{kx}}{h} \\
&= \lim_{h\to 0} e^{kx} \cdot \frac{e^{kh} - 1}{h} \to \; = e^{kx} \cdot \lim_{h\to 0} \frac{a^{kh} - 1}{h} \\
&= ke^{kx} \cdot \lim_{h\to 0} \frac{e^{kh} - 1}{kh} \to \; = ke^{kx} \cdot \ln e = \quad \boxed{ke^{kx}}
\end{aligned}$$

Example

1. $\frac{d}{dx}\left(2^x\right) = 2^x \ln 2.$
2. $\frac{d}{dx}\left(7^x\right) = 7^x \ln 7.$
3. $\frac{d}{dx}\left(e^x\right) = e^x.$
4. $\frac{d}{dx}\left(e^{3x}\right) = 3e^{3x}.$
5. $\frac{d}{dx}\left(e^{-x}\right) = -e^{-x}.$
6. $\frac{d}{dx}\left(e^{-2x}\right) = -2e^{-2x}.$