Section 3.5
Derivative of Trigonometric Functions
2 Lectures

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College of Science

MATHS 101: Calculus I

1. Review of the trigonometric functions (Pre–Calculus).

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2. Limits involving trigonometric functions.

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3. Derivative of the basic trigonometric functions.

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$$\frac{\text{degree}}{180^\circ} \cdot \pi$$

radian $\rightarrow$ degree

$$\frac{\text{radian}}{\pi} \cdot 180^\circ$$

Example

Fill in the following table

degree	0°	30°		60°	90°	120°		180°	270°	360°
Radian			$\frac{\pi}{4}$				$\frac{3\pi}{4}$			

Definition of Sine and Cosine

Unit Circe

Definition of Sine and Cosine

Unit Circe

Definition of Sine and Cosine

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$$-1 \leq \sin\theta, \cos\theta \leq 1$$

Definition of Sine and Cosine

Unit Circe

Right Triangle

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Example

Compute using the unit circle the values of $\sin \frac{\pi}{2}$, $\cos \frac{\pi}{2}$, $\sin \pi$, $\cos \pi$.

Solution:

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$$\sin\frac{\pi}{2} =$$

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Compute using the unit circle the values of $\sin\frac{\pi}{2}$, $\cos\frac{\pi}{2}$, $\sin\pi$, $\cos\pi$.

Solution:

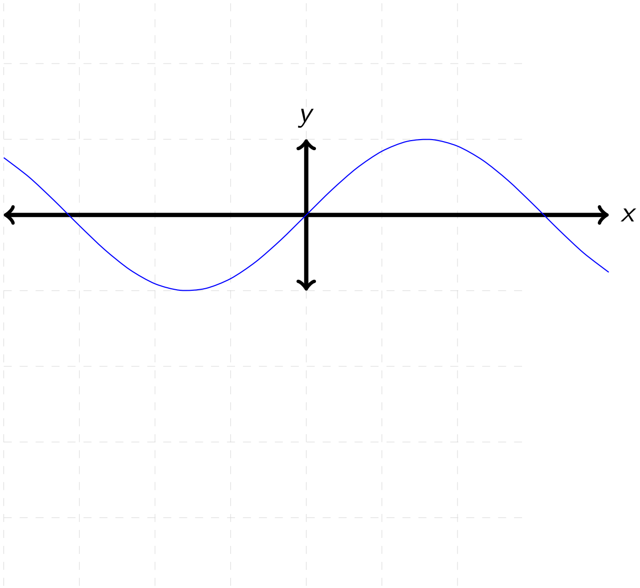
$$\sin\frac{\pi}{2} =$$
$$\cos\frac{\pi}{2} =$$
$$\sin\pi =$$
$$\cos\pi =$$

Example

Fill in the following table using a calculator

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$											
$\cos\theta$											

Graph of since and cosine



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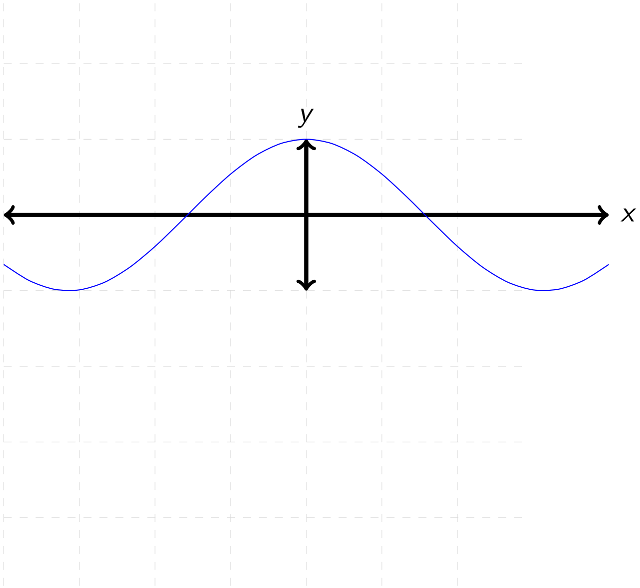

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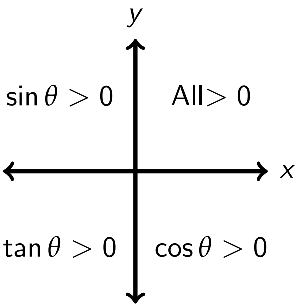

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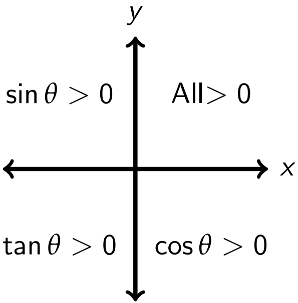

"**A**ll **S**tudents **T**ake **C**alculus"

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   $\cos(a + b) = \cos a\cos b - \sin a\sin b.$
   These are very useful formula!

Continuity of Sine and Cosine

Exercise

Prove that f is continuous at a if and only if

$$\lim_{h \to 0} f(a+h) = f(a)$$

Exercise

Use the trigonometric identities to show that

$$\lim_{h \to 0} \sin(a+h) = \sin(a)$$

and use the exercise above to show that $f(x) = \sin x$ is a continuous function. Do the same for the $f(x) = \cos x$.

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Example

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

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area of $\leq$ area of $\leq$ area of

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Example

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

Solution:

$$\text{area of } \leq \text{area of } \leq \text{area of}$$

$$\frac{1}{2}\sin\theta \leq \frac{1}{2}\theta \leq \frac{1}{2}\tan\theta$$

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Solution:

$$\begin{aligned} \text{area of } &\leq \text{area of } \leq \text{area of} \\ \frac{1}{2}\sin\theta &\leq \frac{1}{2}\theta \leq \frac{1}{2}\tan\theta \\ 1 &\leq \frac{\theta}{\sin\theta} \leq \frac{\tan\theta}{\sin\theta} \end{aligned}$$

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$$\frac{1}{2}\sin\theta \leq \frac{1}{2}\theta \leq \frac{1}{2}\tan\theta$$

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$$1 \geq \frac{\sin\theta}{\theta} \geq \cos\theta \to$$

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1 &\leq \frac{\theta}{\sin\theta} \leq \frac{\tan\theta}{\sin\theta} \\
1 \geq \frac{\sin\theta}{\theta} \geq \cos\theta &\to \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1
\end{aligned}$$

Example

Find

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta}$$

Solution:

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Solution:

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} \cdot \frac{1 + \cos\theta}{1 + \cos\theta}$$

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$$= \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta(1 + \cos\theta)}$$

Example

Find
$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$$

Solution:

$$\begin{aligned}\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}\\ &= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)}\\ &= \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)}\end{aligned}$$

Example

Find

$$\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta}$$

Solution:

$$\begin{aligned}
\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} &= \lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta} \\
&= \lim_{\theta \to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{(1+\cos\theta)}
\end{aligned}$$

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&= \lim_{\theta \to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{(1+\cos\theta)} \\
&= 1 \cdot 0
\end{aligned}$$

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\end{aligned}$$

Example

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Solution:

$$\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta \cos \theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta}$$

Example

Find

$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta}$$

Solution:

$$\begin{aligned}\lim_{\theta\to 0}\frac{\tan\theta}{\theta} &= \lim_{\theta\to 0}\frac{\sin\theta}{\theta\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}\\ &= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{1}{(\cos\theta)}\end{aligned}$$

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\end{aligned}$$

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Solution:

$$\begin{aligned}
\lim_{\theta \to 0} \frac{\tan \theta}{\theta} &= \lim_{\theta \to 0} \frac{\sin \theta}{\theta \cos \theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} \\
&= \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \frac{1}{(\cos \theta)} \\
&= 1 \cdot 1 \\
&= 1
\end{aligned}$$

Three Important limits

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$$

$$\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1$$

Example

Find

$$\lim_{x\to 0} \frac{\sin(7x)}{x}$$

Solution:

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$$\lim_{x \to 0} \frac{\sin(7x)}{x}$$

Solution:

$$\lim_{x \to 0} \frac{\sin(7x)}{x} = \lim_{x \to 0} \frac{7\sin(7x)}{7x}$$

Example

Find

$$\lim_{x \to 0} \frac{\sin(7x)}{x}$$

Solution:

$$\lim_{x \to 0} \frac{\sin(7x)}{x} = \lim_{x \to 0} \frac{7 \sin(7x)}{7x}$$
$$= 7 \lim_{x \to 0} \frac{\sin(7x)}{7x}$$

Example

Find

$$\lim_{x\to 0} \frac{\sin(7x)}{x}$$

Solution:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(7x)}{x} &= \lim_{x\to 0} \frac{7\sin(7x)}{7x}\\ &= 7 \lim_{x\to 0} \frac{\sin(7x)}{7x}\\ &= 7\end{aligned}$$

Example

Find

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$$

Solution:

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$$\lim_{x\to 0} -x \le \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \le \lim_{x\to 0} x$$

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Example

Find
$$\lim_{x \to 0^+} \sqrt{x} e^{\sin\left(\frac{\pi}{x}\right)}$$

Solution:

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Example

Find

$$\lim_{x \to \infty} \frac{\sin(x)}{x^2 + 1}$$

Solution:

Example

Find

$$\lim_{x\to\infty} \frac{\sin(x)}{x^2+1}$$

Solution:

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$$
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\lim_{x\to\infty} 0 &\le \lim_{x\to\infty} \frac{\sin^2(x)}{x^2+1} \le \lim_{x\to\infty} \frac{1}{x^2+1} \\
0 &\le \lim_{x\to\infty} \frac{\sin^2(x)}{x^2+1} \le 0 \\
&\lim_{x\to\infty} \frac{\sin(x)}{x^2+1} = 0
\end{aligned}
$$

Example

For which value(s) of k is the function defined by

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x < 0 \\ 2e^{3x} - k, & x \geq 0 \end{cases}$$

continuous at $x = 0$?

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we have $1 = 2 - k \rightarrow$

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we have $1 = 2 - k \to k = 1$.