Section 3.6 The chain rule 1 Lecture

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MATHS 101: Calculus I

# Motivation

Goal: We want to derive rules to find the derivative of composite of two functions f(g(x))

### Example

We want to find (in a general way) the derivative of the functions (Note the inner and the outer functions)

• 
$$f(x) = (3x^2 + 5x + 1)^3 = (3x^2 + 5x + 1)^3$$
  
•  $f(x) = (2x^3 - 8x)^{\frac{-4}{3}} = (2x^3 - 8x)^{\frac{-4}{3}}$   
•  $f(x) = \frac{4}{x^2 + 5} = 4(x^2 + 5)^{-1} = (x^2 + 5)^{\frac{-1}{3}}$ 

# The Chain Rule

#### Theorem

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

 $(f(g(x)))' = derivative of outer (inner) \cdot (derivative of inner)$ 

Find the derivative of each of the following:

• 
$$f(x) = (3x^2 + 5x + 1)^3$$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.

(1) 
$$f(x) = (3x^2 + 5x + 1)^3 = (3x^2 + 5x + 1)^3$$

 $f'(x) = \text{derivative of outer (inner)} \cdot (\text{derivative of inner})$  $= 3(3x^2 + 5x + 1)^2 \cdot (6x + 5)$ 

Find the derivative of each of the following:  $f(x) = sin(x^3)$ 

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.

 $f(x) = \sin(x^3) = = \sin(x^3).$ 

 $f'(x) = \text{derivative of outer (inner)} \cdot (\text{derivative of inner})$  $= \cos(x^3) \cdot (3x^2)$ 

Find the derivative of each of the following:  $f(x) = \sin^3 x$ 

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.

 $f(x) = \sin^3 x = (\sin x)^3.$ 

 $f'(x) = \text{derivative of outer (inner)} \cdot (\text{derivative of inner})$  $= 3(\sin x)^2 \cdot (\cos x)$ 

Find the derivative of each of the following:  $f(x) = \sqrt[4]{x^2 + \tan x}$ 

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.  $f(x) = (x^2 + \tan x)^{\frac{1}{4}} = (x^2 + \tan x)^{\frac{1}{4}}.$ 

 $f'(x) = \text{derivative of outer (inner)} \cdot (\text{derivative of inner})$  $= \frac{1}{4}(x^2 + \tan x)^{\frac{-3}{4}} \cdot (2x + \sec^2 x)$ 

Find the derivative of each of the following:  $f(x) = \sqrt{x^3 + e^x - \sec x}$ 

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.

 $f(x) = \sqrt{x^3 + e^x - \sec x} = \sqrt{x^3 + e^x - \sec x}.$ 

 $f'(x) = \text{derivative of outer (inner)} \cdot (\text{derivative of inner})$  $= \frac{1}{2\sqrt{x^3 + e^x - \sec x}} \cdot (3x^2 + e^x - \sec x \tan x)$ 

Find the derivative of each of the following:  $f(x) = e^{x^2+2x}$ 

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.  $f(x) = e^{x^2+2x} = e^{x^2+2x}$ .

 $f'(x) = \text{derivative of outer (inner)} \cdot (\text{derivative of inner})$  $= e^{x^2 + 2x} \cdot (2x + 2)$ 

Find the derivative of each of the following:  $f(x) = (x \sin x)^5$ 

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.

 $f(x) = (x \sin x)^5 = (x \sin x)^5.$ 

 $f'(x) = \text{derivative of outer (inner)} \cdot (\text{derivative of inner})$  $= 5(x \sin x)^4 \cdot (\sin x + x \cos x)$ 

# Find the derivative of each of the following: $f(x) = \tan\left(\frac{x+1}{x-1}\right)$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.

 $f(x) = \tan\left(\frac{x+1}{x-1}\right) = \tan\left(\frac{x+1}{x-1}\right).$ 

 $f'(x) = \text{derivative of outer (inner)} \cdot (\text{derivative of inner})$  $= \tan\left(\frac{x+1}{x-1}\right) \cdot \left(\frac{x+1}{x-1}\right)'$  $= \tan\left(\frac{x+1}{x-1}\right) \cdot \left(\frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}\right)$  $= \tan\left(\frac{x+1}{x-1}\right) \cdot \left(\frac{-2}{(x-1)^2}\right)$ 

Find the derivative of each of the following:  $f(x) = \sqrt{x + \sqrt{x}}$ 

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.

 $f(x) = \sqrt{x + \sqrt{x}} = \sqrt{x + \sqrt{x}}.$ 

 $f'(x) = \text{derivative of outer (inner)} \cdot (\text{derivative of inner})$  $= \frac{1}{2\sqrt{x + \sqrt{x}}} \cdot \left(1 + \frac{1}{2\sqrt{x}}\right)$ 

Find the derivative of each of the following: f(x) = sin(sin(sin x))

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.

 $f(x) = \sin(\sin(\sin x)) = = \frac{\sin(\sin(\sin x))}{\sin(\sin x)}.$ 

 $f'(x) = \text{derivative of outer (inner)} \cdot (\text{derivative of inner})$ =  $\cos(\sin(\sin x)) \cdot (\sin(\sin x))'$ =  $\cos(\sin(\sin x)) \cdot (\sin(\sin x))'$ =  $\cos(\sin(\sin x)) \cdot (\cos(\sin x)) \cdot (\cos x)$ 

Find the derivative of each of the following:  $f(x) = \sec^2\left(\frac{1}{x}\right)$ 

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.

 $f(x) = \sec^2\left(\frac{1}{x}\right) = = \frac{\sin(\sin(\sin x))}{\sin(\sin x)}.$ 

$$= 2 \sec\left(\frac{1}{x}\right) \sec\left(\frac{1}{x}\right) \tan\left(\frac{1}{x}\right) \cdot \left(\frac{-1}{x^2}\right)$$

#### Exercise

Find the derivative of  $f(x) = \tan^2(\sin^3 x)$ 

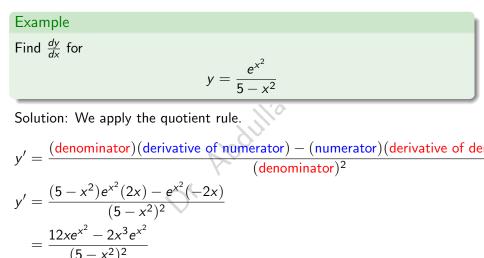


Find the derivative of each of the following:  $f(x) = e^{\sin x} + \sin(e^x)$ 

Solution:

$$f'(x) = e^{\sin x} \cdot \cos x + \cos (e^x) \cdot e^x$$

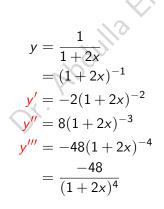
# Chain Rule and Quotient Rule



# (Old Final Exam Question) Find $\frac{d^3y}{dx^3}$ for

$$y = \frac{1}{1+2x}$$

Solution:



Find an equation of the tangent and normal lines to the curve  $y = (x^2 + 1)^3 (2x - 3)^2$  at x = 1.

Solution: Recall that the slope of the tangent line is the derivative. so we have to use the product rule

y'(x) = (derivative of first) (second) + (first) (derivative of second)  $y'(x) = 3(x^2 + 1)^2 (2x)(2x - 3)^2 + (x^2 + 1)^3 (2(2x - 3)(2))$ m = y'(1) = -8

Now  $x_1 = 1$  and  $y_1 = y(1) = 8$ . Hence the equation of the tangent line is

$$y - y_1 = m(x - x_1) \rightarrow y - 8$$
 = -8(x - 1)  
y + 8x = 16

The equation of the normal line is

$$y - y_1 = \frac{-1}{m}(x - x_1) \to y - 8$$
  $= \frac{1}{8}(x - 1)$ 

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Find an equation of the tangent and normal lines to the curve y = sin(sin x) at  $x = \pi$ .

Solution: Recall that the slope of the tangent line is the derivative. so we have to use the product rule

$$y'(x) = \cos(\sin x) \cdot (\cos x)$$
$$m = y'(\pi) = \cos(\sin \pi) \cdot (\cos \pi) = -1$$

Now  $x_1 = \pi$  and  $y_1 = y(1) = 0$ . Hence the equation of the tangent line is

$$y - y_1 = m(x - x_1) \rightarrow y - 0 = -(x - \pi)$$
  
$$y + x = \pi$$

The equation of the normal line is

$$y - y_1 = \frac{-1}{m}(x - x_1) \to y - 0$$
  $= \frac{-1}{-1}(x - \pi)$ 

$$y - x = -\pi$$

## Exercise

Find an equation of the tangent and normal lines of the curve  $y = \sin x + \sin^2 x$  at x = 0.

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# Given the following table

x	f	f'	g	g′
0	1	5	1	$\frac{1}{3}$
1	3	$\frac{-1}{3}$	-4	$\frac{-8}{3}$

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$$(f \circ g)'(0) = f'(g(0)) \cdot g'(0) = f'(1) \cdot \frac{1}{3} = \frac{-1}{9}$$
  

$$(g \circ f)'(0) = g'(f(0)) \cdot f'(0) = g'(1) \cdot 5 = \frac{-40}{3}$$
  

$$(gf)'(0) = g'(0)f(0) + g(0)f'(0) = \frac{1}{3}(1) + 1(5) =$$
  

$$(fg)'(0) = f'(0)g(0) + f(0)g'(0) = (5)(1) + 1(\frac{1}{3}) =$$