Section 3.6
The chain rule
1 Lecture

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MATHS 101: Calculus I

Motivation

Goal: We want to derive rules to find the derivative of composite of two functions $f(g(x))$

Example

We want to find (in a general way) the derivative of the functions (Note the inner and the outer functions)

- $f(x) = (3x^2 + 5x + 1)^3 = (\underbrace{3x^2 + 5x + 1}_{inner})^{\overbrace{3}^{outer}}$
- $f(x) = (2x^3 - 8x)^{\frac{-4}{3}} = (\underbrace{2x^3 - 8x}_{inner})^{\overbrace{\frac{-4}{3}}^{outer}}$
- $f(x) = \frac{4}{x^2+5} = = 4(x^2 + 5)^{-1} = (\underbrace{x^2 + 5}_{inner})^{\overbrace{-1}^{outer}}$

The Chain Rule

Theorem

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

$$(f(g(x)))' = \textit{derivative of outer}\,(\textit{inner}) \cdot (\textit{derivative of inner})$$

Example

Find the derivative of each of the following:

1. $f(x) = (3x^2 + 5x + 1)^3$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.

(1) $f(x) = (3x^2 + 5x + 1)^3 = (3x^2 + 5x + 1)^3$.

$$\begin{aligned} f'(x) &= \text{derivative of outer}\,(\text{inner}) \cdot (\text{derivative of inner}) \\ &= 3(3x^2 + 5x + 1)^2 \cdot (6x + 5) \end{aligned}$$

Example

Find the derivative of each of the following: $f(x) = \sin(x^3)$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.
$f(x) = \sin(x^3) = = \sin(x^3)$.

$$\begin{aligned} f'(x) &= \text{derivative of outer}\,(\text{inner}) \cdot (\text{derivative of inner}) \\ &= \cos(x^3) \cdot (3x^2) \end{aligned}$$

Example

Find the derivative of each of the following: $f(x) = \sin^3 x$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.
$f(x) = \sin^3 x = (\sin x)^3$.

$$
\begin{aligned}
f'(x) &= \text{derivative of outer}\,(\text{inner}) \cdot (\text{derivative of inner}) \\
&= 3(\sin x)^2 \cdot (\cos x)
\end{aligned}
$$

Example

Find the derivative of each of the following:
$f(x) = \sqrt[4]{x^2 + \tan x}$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.
$f(x) = (x^2 + \tan x)^{\frac{1}{4}} = (x^2 + \tan x)^{\frac{1}{4}}$.

$$\begin{aligned} f'(x) &= \text{derivative of outer (inner)} \cdot (\text{derivative of inner}) \\ &= \frac{1}{4}(x^2 + \tan x)^{\frac{-3}{4}} \cdot (2x + \sec^2 x) \end{aligned}$$

Example

Find the derivative of each of the following: $f(x) = \sqrt{x^3 + e^x - \sec x}$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.

$f(x) = \sqrt{x^3 + e^x - \sec x} = = \sqrt{x^3 + e^x - \sec x}.$

$$f'(x) = \text{derivative of outer (inner)} \cdot (\text{derivative of inner})$$

$$= \frac{1}{2\sqrt{x^3 + e^x - \sec x}} \cdot (3x^2 + e^x - \sec x \tan x)$$

Example

Find the derivative of each of the following: $f(x) = e^{x^2+2x}$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.
$f(x) = e^{x^2+2x} = = e^{x^2+2x}$.

$$f'(x) = \text{derivative of outer}\,(\text{inner}) \cdot (\text{derivative of inner})$$
$$= e^{x^2+2x} \cdot (2x+2)$$

Example

Find the derivative of each of the following: $f(x) = (x \sin x)^5$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.
$f(x) = (x \sin x)^5 = (x \sin x)^5$.

$$\begin{aligned} f'(x) &= \text{derivative of outer}\,(\text{inner}) \cdot (\text{derivative of inner}) \\ &= 5(x \sin x)^4 \cdot (\sin x + x \cos x) \end{aligned}$$

Example

Find the derivative of each of the following: $f(x) = \tan\left(\frac{x+1}{x-1}\right)$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.
$f(x) = \tan\left(\frac{x+1}{x-1}\right) = \tan\left(\frac{x+1}{x-1}\right)$.

$$\begin{aligned} f'(x) &= \text{derivative of outer}\,(\text{inner}) \cdot (\text{derivative of inner}) \\ &= \tan\left(\frac{x+1}{x-1}\right) \cdot \left(\frac{x+1}{x-1}\right)' \\ &= \tan\left(\frac{x+1}{x-1}\right) \cdot \left(\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}\right) \\ &= \tan\left(\frac{x+1}{x-1}\right) \cdot \left(\frac{-2}{(x-1)^2}\right) \end{aligned}$$

Example

Find the derivative of each of the following: $f(x) = \sqrt{x + \sqrt{x}}$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.
$f(x) = \sqrt{x + \sqrt{x}} = = \sqrt{x + \sqrt{x}}$.

$$f'(x) = \text{derivative of outer}\,(\text{inner}) \cdot (\text{derivative of inner})$$
$$= \frac{1}{2\sqrt{x + \sqrt{x}}} \cdot \left(1 + \frac{1}{2\sqrt{x}}\right)$$

Example

Find the derivative of each of the following: $f(x) = \sin(\sin(\sin x))$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.

$f(x) = \sin(\sin(\sin x)) = = \sin(\sin(\sin x))$.

$$
\begin{aligned}
f'(x) &= \text{derivative of outer}(\text{inner}) \cdot (\text{derivative of inner}) \\
&= \cos(\sin(\sin x)) \cdot (\sin(\sin x))' \\
&= \cos(\sin(\sin x)) \cdot (\sin(\sin x))' \\
&= \cos(\sin(\sin x)) \cdot (\cos(\sin x)) \cdot (\cos x)
\end{aligned}
$$

Example

Find the derivative of each of the following: $f(x) = \sec^2\left(\frac{1}{x}\right)$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.
$f(x) = \sec^2\left(\frac{1}{x}\right) = = \sin(\sin(\sin x))$.

$$= 2\sec\left(\frac{1}{x}\right)\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)\cdot\left(\frac{-1}{x^2}\right)$$

Exercise

Find the derivative of $f(x) = \tan^2\left(\sin^3 x\right)$

Example

Find the derivative of each of the following: $f(x) = e^{\sin x} + \sin(e^x)$

Solution:

$$f'(x) = e^{\sin x} \cdot \cos x + \cos(e^x) \cdot e^x$$

Chain Rule and Quotient Rule

Example

Find $\frac{dy}{dx}$ for

$$y = \frac{e^{x^2}}{5 - x^2}$$

Solution: We apply the quotient rule.

$$y' = \frac{(\text{denominator})(\text{derivative of numerator}) - (\text{numerator})(\text{derivative of de}}{(\text{denominator})^2}$$

$$y' = \frac{(5 - x^2)e^{x^2}(2x) - e^{x^2}(-2x)}{(5 - x^2)^2}$$

$$= \frac{12xe^{x^2} - 2x^3e^{x^2}}{(5 - x^2)^2}$$

Example

(Old Final Exam Question) Find $\frac{d^3y}{dx^3}$ for

$$y = \frac{1}{1+2x}$$

Solution:

$$\begin{aligned} y &= \frac{1}{1+2x} \\ &= (1+2x)^{-1} \\ y' &= -2(1+2x)^{-2} \\ y'' &= 8(1+2x)^{-3} \\ y''' &= -48(1+2x)^{-4} \\ &= \frac{-48}{(1+2x)^4} \end{aligned}$$

Example

Find an equation of the tangent and normal lines to the curve $y = (x^2+1)^3(2x-3)^2$ at $x = 1$.

Solution: Recall that the slope of the tangent line is the derivative. so we have to use the product rule

$$y'(x) = (\text{derivative of first})\,(\text{second}) + (\text{first})(\text{derivative of second})$$

$$y'(x) = 3(x^2+1)^2(2x)(2x-3)^2 + (x^2+1)^3(2(2x-3)(2))$$

$$m = y'(1) = -8$$

Now $x_1 = 1$ and $y_1 = y(1) = 8$. Hence the equation of the tangent line is

$$y - y_1 = m(x - x_1) \rightarrow y - 8 = -8(x-1)$$

$$y + 8x = 16$$

The equation of the normal line is

$$y - y_1 = \frac{-1}{m}(x - x_1) \rightarrow y - 8 = \frac{1}{8}(x-1)$$

Example

Find an equation of the tangent and normal lines to the curve $y = \sin(\sin x)$ at $x = \pi$.

Solution: Recall that the slope of the tangent line is the derivative. so we have to use the product rule

$$y'(x) = \cos(\sin x) \cdot (\cos x)$$
$$m = y'(\pi) = \cos(\sin \pi) \cdot (\cos \pi) = -1$$

Now $x_1 = \pi$ and $y_1 = y(1) = 0$. Hence the equation of the tangent line is

$$y - y_1 = m(x - x_1) \to y - 0 = -(x - \pi)$$
$$y + x = \pi$$

The equation of the normal line is

$$y - y_1 = \frac{-1}{m}(x - x_1) \to y - 0 = \frac{-1}{-1}(x - \pi)$$
$$y - x = -\pi$$

Exercise

Find an equation of the tangent and normal lines of the curve $y = \sin x + \sin^2 x$ at $x = 0$.

Example

Given the following table

x	f	f'	g	g'
0	1	5	1	$\frac{1}{3}$
1	3	$\frac{-1}{3}$	-4	$\frac{-8}{3}$

$$(f \circ g)'(0) = f'(g(0)) \cdot g'(0) = f'(1) \cdot \frac{1}{3} = \frac{-1}{9}$$

$$(g \circ f)'(0) = g'(f(0)) \cdot f'(0) = g'(1) \cdot 5 = \frac{-40}{3}$$

$$(gf)'(0) = g'(0)f(0) + g(0)f'(0) = \frac{1}{3}(1) + 1(5) =$$

$$(fg)'(0) = f'(0)g(0) + f(0)g'(0) = (5)(1) + 1(\frac{1}{3}) =$$