Section 3.7
Implicit Differentiation
1 Lecture

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MATHS 101: Calculus I

Implicitly Defined Functions

So far, we have seen explicitly defined functions, i.e., functions of the form $y = f(x)$, where y is expressed totally in terms of x. Now we want to deal with implicitly defined functions such as the following:

Example

1. $x^2 + y^2 = 4.$
2. $e^{xy} - x^2 + 4y = 5x - 4.$
3. $\ln(y) + e^{2x} = y^2 e^{-x}.$

The above functions are said to be implicitly defined function. Note that sometimes, it is hard (or even impossible) to write y alone as a function of x.

Goal: To find the derivative y' of implicitly defined functions.

Example

Find y' for

$$y^2 = x$$

Solution 1:
We write y in term of x, so we get $y = \pm\sqrt{x}$. Hence,

$$y' = \pm\frac{1}{2\sqrt{x}}$$

Solution 2: To find the derivative y', we differentiate both sides with respect to x to get

$$2yy' = 1$$

$$y' = \frac{1}{2y}$$

Exercise

Find $\frac{dy}{dx}$ for $xy = 1$ by two ways.

Example

Find y' for

$$x^2 + y^2 = 4$$

and use it to find an equation of the tangent line at $(1, \sqrt{3})$.

Solution: To find the derivative y', we differentiate both sides with respect to x to get

$$2x + 2yy' = 0$$

$$2yy' = -2x \to y' = \frac{-2x}{2y}$$

$$y' = \frac{-x}{y}$$

$$m = y'_{|(1,\sqrt{3})} = \frac{-1}{\sqrt{3}}$$

Hence the equation of the tangent line is

$$y - y_1 = m(x - x_1) \to y - \sqrt{3} = \frac{-1}{\sqrt{3}}(x - 1) \to \sqrt{3}y + x = 4$$

Example

Find y' for

$$xy^2 - 6x = 5 + 2y$$

Solution: To find the derivative y', we differentiate both sides with respect to x to get

$$\begin{aligned} y^2 + 2xyy' - 6 = 2y' &= 0 \\ 2xyy' - 2y' &= -y^2 + 6 \\ (2xy - 2)y' &= -y^2 + 6 \\ y' &= \frac{-y^2 + 6}{2xy - 2} \end{aligned}$$

Example

Find y' for

$$x + e^y = y + e^x$$

Solution: To find the derivative y', we differentiate both sides with respect to x to get

$$\begin{aligned} 1 + e^y \cdot y' &= y' + e^x \\ e^y y' - y' &= e^x - 1 \\ (e^y - 1)y' &= e^x - 1 \\ y' &= \frac{e^x - 1}{e^y - 1} \end{aligned}$$

Example

(Old Final Exam Question) Find y' for

$$xy = \cot(y)$$

Solution: To find the derivative y', we differentiate both sides with respect to x to get

$$\begin{aligned} y + x \cdot y' &= -\csc^2(y) \cdot y' \\ x \cdot y' + \csc^2(y) \cdot y' &= -y \\ (x + \csc^2(y))y' &= -y \\ y' &= \frac{-y}{(x + \csc^2(y))} \end{aligned}$$

Exercise

Find $\frac{dy}{dx}$ for $\sin(xy) = \frac{1}{2}$.

Example

(The Folium of Descarte')

(a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find an equation of the tangent line at $(3, 3)$.

(c) Find an equation of the normal line at $(3, 3)$.

Solution: (a) To find the derivative y', we differentiate both sides with respect to x to get

$$\begin{aligned} 3x^2 + 3y^2y' &= 6y + 6xy' \\ 3y^2y' - 6xy' &= 6y - 3x^2 \\ (3y^2 - 6x)y' &= 6y - 3x^2 \\ y' &= \frac{6y - 3x^2}{3y^2 - 6x} \end{aligned}$$

Continue...

Example

(The Folium of Descarte')

(a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find an equation of the tangent line at $(3, 3)$.

(c) Find an equation of the normal line at $(3, 3)$.

Solution: (b) The equation of the tangent line is given by

$$y - y_1 = m(x - x_1)$$

We find the slope at $(3, 3)$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x}$$

$$m = \frac{6(3) - 3(3)^2}{3(3)^2 - 6(3)}$$

$$m = -1$$

Continue...

Hence we have

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 3 &= -(x - 3) \\
y + x &= 6
\end{aligned}
$$

Example

Find an equation of the tangent and normal lines at $(0, \pi)$ for the curve

$$x^2 \cos^2 y - \sin y = 0$$

Solution: To find the derivative y', we differentiate both sides with respect to x to get

$$2x \cos^2 y + 2x^2 \cos y \sin y \cdot y' - \cos y \cdot y' = 0$$

$$2(0) \cos^2(\pi) + 2(0)^2 \cos(\pi) \sin(\pi) \cdot m - \cos(\pi) \cdot m = 0 \to m = 0$$

Hence the equation of the tangent line is

$$y - y_1 = m(x - x_1) \to y - \pi = 0(x - 1) \to \boxed{y = \pi} \text{ horizontal}$$

Hence the equation of the normal line is

$$\boxed{x = 0} \text{ vertical}$$

Example

Find $\frac{d^2y}{dx^2}$ for

$$xy + y - x = 4$$

Solution:

$$y + xy' + y' - 1 = 0 \rightarrow y' = \frac{1-y}{x+1}$$

$$y' + y' + xy'' + y'' = 0$$

$$(x+1)y'' = -2y'$$

$$y'' = \frac{-2y'}{x+1}$$

$$y'' = \frac{-2\frac{1-y}{x+1}}{x+1}$$

$$y'' = \frac{-2(1-y)}{(x+1)^2}$$