Section 3.8
Derivative of the inverse function and logarithms
3 Lecture

Dr. Abdulla Eid

College of Science

MATHS 101: Calculus I

1. Inverse Functions (1 lecture).
2. Logarithms.
3. Derivative of the inverse function (1 lecture).
4. Logarithmic differentiation (1 lecture).

2- Logarithmic Function

Consider the exponential function $f(x) = a^x$.
Question: Does $f(x)$ has an inverse? Why?
Answer: Yes, by the horizontal line test.

- $f^{-1}(x)$ is called **logarithmic function** base a and it is denoted by

$$f^{-1}(x) = \log_a x$$

Note: (The fundamental equations)

1. $f(f^{-1})(x) = x$, so we have $a^{\log_a x} = x$.
2. $f^{-1}(f(x)) = x$, so we have $\log_a a^x = x$.

$$\underbrace{\log_a x = y}_{\text{logarithmic form}} \quad \text{if and only if} \quad \underbrace{x = a^y}_{\text{exponential form}}$$

If $a = e = 2.718281828\ldots$ (Euler number), then we simply write $\log_e$ as ln "ell en" and it is called the **natural logarithm**.

Properties of Logarithms

1. $\log_a(m \cdot n) = \log_a m + \log_a n$.
2. $\log_a(\frac{m}{n}) = \log_a m - \log_a n$.
3. $\log_a m^r = r \log_a m$.
4. $\log_a 1 = 0$.
5. $\log_a a = 1$.
6. (change of bases) $\log_a m = \frac{\log_b m}{\log_b a}$.

Exercise

Use the fundamental equations to prove these six properties of the logarithms.

Example

(Expansion) Write the following expression as sum or difference of logarithms

1. $\ln(\frac{x}{wz^2}) = \ln x - \ln(wz^2) = \ln x - (\ln w + \ln z^2) = \ln x - \ln w - 2\ln z.$
2. $\ln(\frac{x+1}{x+5})^4 = 4\ln(\frac{x+1}{x+5}) = 4(\ln(x+1) - \ln(x+5)).$
3. $\ln(\frac{\sqrt{x}}{(x^2)(x+3)^4}) = \ln\sqrt{x} - \ln x^2 - \ln(x+3)^4 =$ $\ln x^{\frac{1}{2}} - 2\ln x - 4\ln(x+3) = \frac{1}{2}\ln x - 2\ln x - 4\ln(x+3) =$ $-\frac{3}{2}\ln x - 4\ln(x+3).$

Exercise

Write each of the following expression as sum or difference of logarithms:

(1) $\log_3(\frac{5\cdot 7}{4})$ (2) $\log_2(\frac{x^5}{y^2})$ (3) $\log(\frac{x^2 z}{wy^2})$ (4) $\ln\sqrt{\frac{x+1}{x-2}}$.

Example

Write each of the following logarithm in terms of natural logarithm.

1. $\log_3 x = \frac{\ln x}{\ln 3}$.
2. $\log_6 7 = \frac{\ln 7}{\ln 6}$.
3. $\log_2 y = \frac{\ln y}{\ln 2}$.

The derivative of the inverse function

Strategy:

Goal: We want to find $\frac{d}{dx}\left(f^{-1}(x)\right)$.

$$
\begin{aligned}
\text{Write } y = f^{-1}(x),&\ \text{ we want to find } y' \\
f(y) &= f(f^{-1}(x)) \\
f(y) &= x \\
f'(y) \cdot y' &= 1 \\
y' &= \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}
\end{aligned}
$$

Geometric Interpretation *

Note that

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$

so the slope of f^{-1} is reciprocal to the slope of f. Geometrically,

Example

Let $f(x) = x^3 - 3x^2 - 1$. Find $\frac{d}{dx}(f(x))$ and $\frac{d}{dx}\left(f^{-1}(x)\right)$ at the point $(3, -1)$

Solution:

$$\frac{d}{dx}(f(x)) = 3x^2 - 6x$$

$$\frac{d}{dx}(f(x))_{(3,-1)} = 3(3)^2 - 6(3) = 9$$

$$\begin{aligned}\frac{d}{dx}\left(f^{-1}(x)\right) &= \frac{1}{f'(y)} \\ &= \frac{1}{3y^2 - 6y}\end{aligned}$$

$$\frac{d}{dx}\left(f^{-1}(x)\right)_{(3,-1)} = \frac{1}{3(3)^2 - 6(3)} = \frac{1}{9}$$

Derivative of ln

Example

Find $\frac{d}{dx}(\ln x)$.

Solution:

$$
\begin{aligned}
y &= \ln x \\
e^y &= x \\
e^y \cdot y &= 1 \\
y &= \frac{1}{e^y} \\
&\boxed{y' = \frac{1}{x}}
\end{aligned}
$$

Exercise

Find y' if $y = \log_a x$. (Hint: Use the change of base formula to change it to ln)

Recall

The Chain Rule

Theorem

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

$$(f(g(x)))' = \textit{derivative of outer}\,(\textit{inner}) \cdot (\textit{derivative of inner})$$

Example

Find y' for each of the following:

1. $f(x) = \ln x^2 = \ln x^2 \rightarrow y' = \frac{1}{x^2} \cdot 2x.$
2. $f(x) = \ln(2x+3) = \ln(2x+3) \rightarrow y' = \frac{1}{(2x+3)} \cdot 2.$
3. $f(x) = x \ln x \rightarrow y' = \ln x + x \cdot \frac{1}{x} = \ln x + 1.$
4. $f(x) = \ln(\ln x) = \ln(\ln x) \rightarrow y' = \frac{1}{(\ln x)} \cdot \frac{1}{x}.$
5. $f(x) = \ln(\sin x) = \ln(\sin x) \rightarrow y' = \frac{1}{(\sin x)} \cdot \cos x = \cot x.$
6. $f(x) = \sin(\ln x) = \sin(\ln x) \rightarrow y' = \cos(\ln x) \frac{1}{(x)}.$

Derivative using the properties of Logarithms

Example

Find the derivative of

1. $f(x) = \ln x^{2016}$

Solution: First we re–write the function in terms using the properties of the ln to get a simplified function:

$$f(x) = 2016 \ln x$$

Hence

$$f'(x) = 2016 \frac{1}{x}$$

Exercise

Using the chain rule, find the derivative of the function of the previous example *without using the properties of the* ln, i.e., find $f'(x)$ for

$$f(x) = \ln(x^{2016})$$

Derivative using the properties of Logarithms

Example

Find the derivative of

1. $f(x) = \ln \sqrt[3]{\frac{x^3-1}{x^3+1}}$

Solution: First we re–write the function in terms using the properties of the ln to get a simplified function:

$$\begin{aligned} f(x) &= \ln\left(\frac{x^3-1}{x^3+1}\right)^{\frac{1}{3}} \\ &= \frac{1}{3}\left(\ln(x^3-1) - \ln(x^3+1)\right) \end{aligned}$$

Continue...

We write the inner function in blue and the outer function in red and we apply the chain rule.

$$\text{derivative of outer}\,(\text{inner}) \cdot (\text{derivative of inner})$$

$$f(x) = \frac{1}{3}\left(\ln(x^3 - 1) - \ln(x^3 + 1)\right)$$
$$f'(x) = \frac{1}{3}\left(\frac{1}{x^3 - 1} \cdot (3x^2) - \frac{1}{x^3 + 1} \cdot (3x^2)\right)$$

Exercise

Using the chain rule, find the derivative of the function of the previous example *without using the properties of the* ln, i.e., find $f'(x)$ for

$$f(x) = \ln\left(\sqrt[3]{\frac{x^3-1}{x^3+1}}\right)$$

Example

Find $\frac{d^4y}{dx^4}$ for

$$y = 5\ln x$$

Solution:

$$\begin{aligned}
y' &= 5\frac{1}{x} \\
&= 5x^{-1} \\
y'' &= -5x^{-2} \\
y''' &= 10x^{-3} \\
y^{(4)} &= -30x^{-4} \\
&= \frac{-30}{x^4}
\end{aligned}$$