Preliminaries
$2\frac{1}{2}$ Lectures

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MATHS 101: Calculus I

Pre Calculus $\rightarrow$ MATHS 101: Calculus

- MATHS 101 is all about functions!
- MATHS 101 is an introductory course to a branch of Mathematics called Calculus.

Calculus

Differentiation

- We want to find the derivative of a function, which is finding the slope of the tangent line to the graph of a function at a given point.

Integration

- We want to find the integrate a function, which is finding the area under the graph of a function on a given interval.

Questions

Question 1 What is the relation between differentiation and integration? In other words, what is the relation between finding the slope of the tangent line and finding the area under the curve of a function?

Question 2 Why they are given together at the same course while they might look as two different branches of mathematics? (one measures the slope and the other measures the area)?

Answer The connection is given in the fundamental theorem of calculus which states (informally) that differentiation and integration are reversing each other! (In fact, both can be defined in terms of a limit!)

In MATHS 101, we will study

1. Limit of a function.
2. Derivative and its applications.
3. Integration and its applications.

Note: We want to differentiate (integrate) all kind of functions. So in MATHS 101, the strategy will be

1. Find the derivative (integral) of the basic functions, e.g., x^n, c, e^x, a^x, $\ln x$, $\log_a x$, $\sin x$, $\cos x$, $\tan x$, $\sin^{-1} x$, etc.
2. Establish rules to find the derivative (integral) of the new functions from the basic ones, i.e., rules for the sum, difference, product, quotient, composite, inverse, etc.

Hope you will have a nice course

Preliminaries: (From High school)

In this lecture, we will go over some important topics from high school. These are

1. Functions.
2. Graphs.
3. Lines.
4. Factoring.

1. Definition of a function

A **function** from a set X to a set Y is an *assignment* (*rule*) that tells how one element x in X is related to only one element y in Y.

Notation:

- $f : X \to Y$.
- $y = f(x)$.   "f of x".
- x is called the input (independent variable) and y is called the output (dependent variable).
- The set X is called the domain and Y is called the co–domain. While the set of all outputs is called the range.

Think about the function as a vending machine!

Question: How to describe a function mathematically?

Answer: By using algebraic formula!

Example

Consider the function

$$f : (-\infty, \infty) \to (-\infty, \infty)$$
$$x \mapsto 3x + 1$$

or simply by $f(x) = 3x + 1$

- f(1)=3(1)+1=4.
- f(0)=3(0)+1=1.
- f(-2)=3(-2)+1=-5.
- f(-7)=3(-7)+1=-20.

- Domain $= (-\infty, \infty)$.
- Co–domain$=(-\infty, \infty)$.
- Range$=(-\infty, \infty)$.

Example

$$f : (-\infty, \infty) \to (-\infty, \infty)$$
$$x \mapsto x^2$$

or simply by $f(x) = x^2$

- f(1)=$(1)^2$=1.
- f(0)=$(0)^2$=0.
- f(-1)=$(-1)^2$=1.
- f(-2)=$(-2)^2$=4.
- f(-4)=$(-4)^2$=16.
- f(4)=$(4)^2$=16.

- Domain $= (-\infty, \infty)$.
- Co–domain$=(-\infty, \infty)$.
- Range$=[0, \infty)$.

Example

$$f : (-\infty, \infty) \to (-\infty, \infty)$$
$$x \mapsto \frac{1}{x}$$

or simply by $f(x) = \frac{1}{x}$

- f(1)=$\frac{1}{1}$=1.
- f(-1)=$\frac{1}{-1}$=-1.
- f(2)=$\frac{1}{2}$=$\frac{1}{2}$.
- f(-4)=$\frac{1}{-4}$=$\frac{-1}{4}$.
- f(100)=$\frac{1}{100}$=$\frac{1}{100}$.
- f(0)=$\frac{1}{0}$=undefine (Problem, so we have to exclude it from the domain!)

- Domain $= \{x \,|\, x \neq 0\}$.
- Co–domain=$(-\infty, \infty)$.
- Range=$\{y \,|, y \neq 0\}$.

Finding Function Values

Recall

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

Example

Let $g(x) = x^2 - 2$. Find

- f(2)=$(2)^2 - 2$=2. (we replace each x with 2).
- f(u)=$(u)^2 - 2$=$u^2 - 2$.
- f(u^2)=$(u^2)^2 - 2$=$u^4 - 2$.
- f($u + 1$)=$(u + 1)^2 - 2$=$u^2 + 2u + 1 - 2 = u^2 + 2u - 1$.

Exercise

Let $f(x) = \frac{x-5}{x^2+3}$. Find

- $f(5)$.
- $f(2x)$.
- $f(x+h)$.
- $f(-7)$.

Example

Let $f(x) = x^2 + 2x$. Find $\frac{f(x+h)-f(x)}{h}$.

Solution:

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2 + 2(x+h) - (x^2+2x)}{h} \\
&= \frac{x^2 + 2xh + h^2 + 2x + 2h - x^2 - 2x}{h} \\
&= \frac{2xh + h^2 + 2h}{h} \\
&= \frac{h(2x+h+2)}{h} \\
&= 2x + h + 2
\end{aligned}$$

Exercise

Let $f(x) = 2x^2 - x + 1$. Find $\frac{f(x)-f(2)}{x-2}$.

2- The graph of a function

Example

Graph (sketch) the function $y = x^2 - 1$.

We substitute values of x to find the values of y and we fill the table

x	-2	-1	0	1	2	3
y						

Note:

- In ideal world, we will need to plot infinitely many points to get a perfect graph, but this is not possible, so our concern is only on the "general shape" of the function by joining only several points by a smooth curve whenever possible.
- In MATHS101, we will be able to graph more complicated functions in an easier way! (using calculus).

3 - Special functions

- $f(x) = c$ is called the constant function. The output is always the constant c and its graph is a horizontal line $y = c$.
- $f(x) = ax + b$ is called the linear function. The graph is always a straight line.
- $f(x) = ax^2 + bx + c$ is called the quadratic function. The graph is always a parabola.
- $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$ is called a polynomial in x.
- $f(x) = \frac{p(x)}{q(x)}$, where $p(x), q(x)$ are polynomials is called the rational function.
- $f(x) = \sqrt[n]{x}$, is called the root function.

Definition

A function f is called an algebraic function if it can be constructed using algebraic operations $(+, -, \cdot, \div, \sqrt[n]{\ })$ starting from polynomials.

Example: $g(x) = \frac{x^4 - x^2 + 1}{x + \sqrt[3]{x}} + (x+1)\sqrt{x+3}$ is an algebraic function.

Transcendental Functions

Definition

A function f is called an transcendental function if it is not algebraic.

These are transcendental functions:

- $f(x) = a^x$ is called the exponential function.
- $f(x) = \log_a x$ is called the logarithmic function.
- $f(x) = \sin x, \cos x, \tan x, \sec x, \cot x, \csc x$ are called the trigonometric function.
- $f(x) = \ln x$ is called the natural logarithmic function where $a = e = 2.71818182\ldots$.

Note: This course is early transcendental calculus course, meaning, we will study all those function right from the beginning.

Example

(Piecewise defined Functions)

$$g(x) = \begin{cases} x-1, & x \geq 3 \\ 3-x^2, & x < 3 \end{cases}$$

- $g(1)=3-(-1)^2=2$.
- $g(-2)=3-(-2)^2=-1$.
- $g(6)=6-1=5$.
- $g(4)=4-1=3$.
- $g(3)=3-1=2$.

Example

(Absolute Value Functions) Let $f(x) = |x|$ be the absolute value function. It can be written as piecewise function as follows:

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

- f(1)=1.
- f(-2)=2.
- f(-6)=6.
- f(0)=0.
- f(-3)=3.

Exercise

Sketch the graph of the absolute value function.

Lines

Recall that the equation of the line is given by $f(x) = ax + b$.

1 - The slope of a line

1. The **slope** of a line is a number that measures how sloppy the line is (how hard to climb the stairs!).

1. Consider the two lines L_1 and L_2 (both of positive slope),but you can see that L_1 has slope greater than L_2.

2. Slope has a clear relation with the angle between the line and the x–axis. if the slope rises, then θ rises too!.

$$\text{Slope} = m = \tan\theta!$$

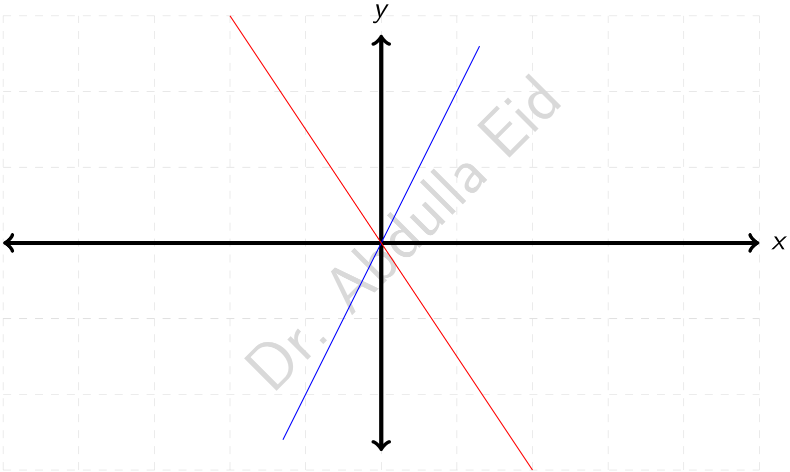
Finding the slope of a line

1. From the equation of the line: Solve the equation for y, i.e., let y be alone. Then, you get
$$y = mx + b$$
and the slope is m.
2. From the graph of the line: Choose any two points (x_1, y_1) and (x_2, y_2) on the line. Then,
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Vertical change}}{\text{Horizontal change}}$$

Special Case: The vertical line **has no** slope. Why?

Geometric Interpretation of the slope

Find the slope of the the blue line and the red line.



For the blue line, every 1 step to the right, we go 2 steps upward.
For the red line, every 2 step to the right, we go 3 steps downward.

2 - Equation of the line

To get the equation of a line, you need to find

- One point on the line (x_1, y_1) and
- The slope of the line m.

Then, the equation of the line is

$$\boxed{y - y_1 = m(x - x_1)} \qquad - - - \qquad \text{“point–slope form“}$$

Other forms:

General Linear Form $ax + by + c = 0$, where a, b, and c have **no** common factor.

Slope–Intercept Form $y = mx + b$, where m is the slope of the line and $(0, b)$ is the y–intercept.

Special Case: The equation of the vertical line is $\boxed{x = x_1}$.

3 - Parallel and Perpendicular Lines

Definition

- Two lines are parallel if

$$m_1 = m_2$$

- Two lines are perpendicular if

$$m_1 m_2 = -1 \text{ or } m_2 = \frac{-1}{m_1}$$

Factoring

1- Factoring by taking common factor:

- $3x + 6 = 3(x + 2)$.
- $x^2 + 6x = x(x + 6)$.
- $x^4 - 2x^3 + 8x^2 = x^2(x^2 - 2x + 8)$.
- $6x^4 + 12x^2 + 6x = 6x(x^3 + 2x + 1)$.
- $7x^5 - 7 = 7(x^5 - 1)$.

2- Factoring by grouping (works well if we have 4 terms):

- $3x^4 + 3x^3 + 7x + 7 = 3x^3(x + 1) + 7(x + 1) = (x + 1)(3x^3 + 7)$.
- $16x^3 - 28x^2 + 12x - 21 = 4x^2(4x - 7) + 3(4x - 7)$
  $= (4x - 7)(4x^2 + 3)$.
- $3xy + 2 - 3x - 2y = 3x(y - 1) + 2(1 - y) =$
  $3x(y - 1) - 2(y - 1) = (3x - 2)(y - 1)$.
- $4y^4 + y^2 + 20y^3 + 5y = y(4y^3 + y + 20y^2 + 5) =$
  $y(y(4y^2 + 1 + 5(4y^2 + 1)) = y(4y^2 + 1)(y + 5)$

Factoring Trinomial

Definition

A *trinomial* is an expression of the form $ax^2 + bx + c$.

To factor such a trinomial, we will use the quadratic formula of to get

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$

where α and β are the solution you will get from the quadratic formula.

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

Factor $8x^2 - 22x + 5$

Solution: Here we have $a = 8, b = -22, c = 5$, so we apply the quadratic formula to find α, β, so we have

$$\alpha, \beta = \frac{1}{4}, \frac{5}{2}$$

Hence

$$\begin{aligned} 8x^2 - 22x + 5 &= 8(x - \frac{1}{4})(x - \frac{5}{2}) \\ &= 8\frac{(4x-1)}{4}\frac{(2x-5)}{2} \\ &= (4x-1)(2x-5) \end{aligned}$$

Exercise

Factor each of the following trinomial expression completely:

1. $2x^2 + 13x - 7$
2. $3x^2 + 11x + 6$
3. $x^2 - 4$
4. $4x^2 - 25$
5. $-6x^2 - 13x + 5$
6. $x^2 + 12x + 36$

Solution:

1. $2x^2 + 13x - 7 = (2x - 1)(x + 7)$.
2. $3x^2 + 11x + 6 = (3x + 2)(x + 3)$.
3. $x^2 - 4 = (x - 2)(x + 2)$.
4. $4x^2 - 25 = (2x - 5)(2x + 5)$.
5. $-6x^2 - 13x + 5 = -(3x - 1)(2x + 5)$.
6. $x^2 + 12x + 36 = (x + 6)(x + 6)$

Factoring Cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2$$

Example

- $x^3 - 8 = x^3 - 2^3 = (x - 2)(x^2 + 2x + 4)$.
- $x^3 + 1 = x^3 + 1^3 = (x + 1)(x^2 - x + 1)$.
- $64x^3 - 1 = 4^3x^3 - 1^3 = (4x - 1)(16x^2 + 4x + 1)$.

Factoring higher degree

$$a^n - b^n = (a-b)(\underbrace{a^{n-1} + a^{n-2}b^2 + a^{n-3}b^2 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1}}_{n\text{ – terms}})$$

Example

- $x^5 - 1 = x^5 - 1^5 = (x-1)(x^4 + x^3 + x^2 + x + 1)$.
- $x^7 + 1 = x^7 - (-1)^7 = (x-1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)$.
- $x^6 - 32 = x^6 - (2)^6 = (x-2)(x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32)$.