Section 3.10
Related Rates
2 Lecture

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MATHS 101: Calculus I

Rate of change

Recall:

- If a line has a slope $m = 3$, it means that for every one step to the right, we move along the line 3 steps up. In this case, as x increases, y increases.
- If a line has a slope $m = -2$, it means that for every 1 step to the right, we move along the line 2 steps down. In this case, as x increases, y decreases.

For general function $y = f(x)$, for every step to the right, how many steps to go up/down? How do we measure that change in y?
If x changes by 1, an estimate of the change in y is $\frac{dy}{dx}$.

Definition

The derivative of $y = f(x)$ can be interpreted as *rate of change* of y in term of x.

In this section, we have our variables are functions in t. so for example, we have

1. $\frac{dx}{dt}$ is the rate change in x with respect to the time t. If $\frac{dx}{dt} = 4$, it means that for every one unit in time, x increases by 4 units.
2. $\frac{dr}{dt}$ is the rate change in r with respect to the time t. If $\frac{dr}{dt} = -2$ cm/second, it means that for every one second, r decreases 2 cm.

Example

If $y = x^2$ and $\frac{dx}{dt} = 3$. What is $\frac{dy}{dt}$ when $x = -1$?

Solution:
Given: $\frac{dx}{dt} = 3$ and $x = -3$.
Required: $\frac{dy}{dt}$.
Relation: $y = x^2$

$$\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dy}{dt} = 2(-1)(3) = -6$$

Example

Air is being pumped in a spherical balloon so that its volume increases at a rate of 500 cm^3/second. How fast is the radius of the balloon increases when the diameter is 50 cm? (Note that the volume of the sphere is $V = \frac{4}{3}\pi r^3$.

Solution:

Given: $\frac{dV}{dt} = 500$ and $D = 50 \to rr = \frac{D}{2} = 25$.

Required: $\frac{dr}{dt}$.

Relation: $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$$

$$500 = 4\pi(25)^2\frac{dr}{dt} \to \frac{dr}{dt} = \frac{500}{4\pi(25)^2} \text{ cm / second}$$

Example

Each side of a square is increasing at a rate of 6 cm/second. At what rate is the area of the square increases when the area of the square is 16 cm^2.

Solution:

Given: $\frac{dx}{dt} = 6$ and $A = 16$.

Required: $\frac{dA}{dt}$.

Relation: $A = x^2$

$$\frac{dA}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dA}{dt} = 2(4)(6) = \frac{dA}{dt} = 24 \text{ cm}^2 \text{ / second}$$

Exercise

The length ℓ of a rectangle is decreasing at the rate of 2 cm/second while the width w is increasing at the rate of 2 cm/second when $\ell = 12$ cm and $w = 5$ cm. Find the rate of change of

1. Area
2. Perimeter
3. Lsength of the diagonal

Example

A ladder is 5 m long rests against a vertical wall. If the bottom of the ladder moves away at rate 0.5 m/second. How fast the top of the ladder slides down when the bottom of the ladder is 4 m from the wall.

Solution:

Given: $\frac{dx}{dt} = 0.5$ and $\ell = 5$, $x = 4$.

Required: $\frac{dy}{dt}$.

Relation: $x^2 + y^2 = 25$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2(4)(0.5) + 2(3)(\frac{dy}{dt}) = 0$$

$$4 + 6\frac{dy}{dt} = 0 \rightarrow \frac{dy}{dt} = \frac{-4}{6} \text{ m / second}$$

Example

An airplane is moving parallel to the ground on elevation 10 km with a rate of 6 km/minute. Find the rate of change for the angle where an observer observes the airplane when his angle is $\frac{\pi}{3}$.

Solution:
Given: $\frac{dx}{dt} = 6$ and $y = 10$, $\theta = \pi/3$.
Required: $\frac{d\theta}{dt}$.
Relation: $\tan\theta = \frac{10}{x}$ (hence $x = \frac{10}{\tan(\frac{\pi}{3})}$)

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{-10}{x^2} \cdot \frac{dx}{dt}$$

$$2\frac{d\theta}{dt} = \frac{-10}{(\,)^2} \cdot 6 \rightarrow \frac{d\theta}{dt} = \text{ radian / minute}$$

Example

A cube's surface area increases at a rate of 72 in^2/second. At what rate the cube volume changes when the edge of the cube is 3 in? (Note that the volume of the cube is $V = x^3$ and the suface area is $S = 6x^2$)

Solution:
Given: $\frac{dS}{dt} = 72$ and $x = 3$.
Required: $\frac{dV}{dt}$.
Relation: $V = x^3$

$$\frac{dV}{dt} = (3x^2)\frac{dx}{dt} \rightarrow \frac{dV}{dt} = 27\frac{dx}{dt}$$

$$S = 6x^2 \rightarrow \frac{dS}{dt} = 12x\frac{dx}{dt} \rightarrow 72 = 36\frac{dx}{dt} \rightarrow \frac{dx}{dt} = 2$$

$$\frac{dV}{dt} = 27(2) = 54 \text{ in}^3/\text{second}$$