Section 3.11
Linear Approximation and Differentials
1.5 Lectures

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MATHS 101: Calculus I

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In this section, we will study:

1. Linear Approximation of a function.
2. Differentials
3. Newton's Method

1 - Linear Approximation

Idea: Given a function f, a number a, and a number x very close to a such that

$$f(a), f'(a) \text{ can be easily computed}$$
$$f(x) \text{ is difficult to compute}$$

Goal: We use $f(a), f'(a)$ to approximate the value $f(x)$.

Example

Consider $f(x) = \sqrt{x}$, $a = 16$, $x = 16.01$. Then

$$f(16) = \sqrt{16} = 4 \quad \text{easy}$$
$$f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8} \quad \text{easy}$$
$$f(16.01) = \sqrt{16.01} = ?? \quad \text{difficulut}$$

Linear Approximation

Given a function f, a number a, and a number x very close to a, we have

$$\begin{aligned} f(x) &\sim \text{Tangent line at } a \\ &\sim y_1 + m(x - x_1) \\ &\sim f(a) + f'(a)(x - a) \end{aligned}$$

Definition

The **linear approximation** of f at a is given by

$$f(a) + f'(a)(x - a)$$

Example

Find the linear approximation of $f(x) = \sqrt{x}$ at $a = 16$ and then use it to approximate $\sqrt{16.01}$.

Solution: We need to find $f'(x)= \frac{1}{2\sqrt{x}}$. Then we have the linear approximation is given by

$$\begin{aligned} f(x) &\sim f(a) + f'(a)(x-a) \\ &\sim \sqrt{16} + \frac{1}{2\sqrt{16}}(x-16) \\ &\sim 4 + \frac{1}{8}(x-16) = 4 + \frac{1}{8}x - 2 \\ &\sim 2 + \frac{1}{8}x \end{aligned}$$

$$\sqrt{16.01} \sim 2 + \frac{1}{8}(16.01) \sim 4.00125$$

Exercise

Compare the answer above with the one you would get if you use a

Example

Find the linear approximation of $f(x) = \sqrt[3]{8-x}$ at $a = 0$.

Solution: We need to find $f'(x) = \frac{-1}{3}(8-x)^{\frac{2}{3}}$. Then we have the linear approximation is given by

$$\begin{aligned} f(x) = \sqrt[3]{8-x} &\sim f(a) + f'(a)(x-a) \\ &\sim \sqrt[3]{8-0} + \frac{-1}{3}(16-0)^{\frac{2}{3}}(x-0) \\ &\sim 2 - \frac{4}{3}(x) \\ &\sim 2 - \frac{4}{3}x \end{aligned}$$

Exercise

Find the linear approximation of $f(x) = \ln(x+1)$ at $a = 0$.

Example

Use linear approximation to approximate the value of $\sin 0.03$.

Solution: Here we know that $x = 0.03$, we need to find the function f and a value a near x that we can compute $f(a), f'(a)$ easily.

Let $f(x) = \sin x$ ($f'(x) = \cos x$) and $a = 0$

$$
\begin{aligned}
f(x) = \sin x &\sim f(a) + f'(a)(x - a) \\
&\sim \sin 0 + \cos 0(x - 0) = 0 + 1(x) \\
\sin x &\sim x \\
\sin 0.03 &\sim 0.03
\end{aligned}
$$

Exercise

Compare the answer above with the one you would get if you use a calculator or a computer.

Exercise

Use linear approximation to approximate the value of $\sin 0.03$. (Hint: Use Exercise 6

2 - Differentials

Definition

Let $y = f(x)$, then the differential dy is given by

$$dy = f'(x)dx$$

Geometric Interpretation:
What is dx and dy?

Example

Find the differential of each of the following functions:

1. $y = 1 + 2x^3 \to dy = 6x^2\, dx$
2. $y = e^x + 4 \to dy = e^x\, dx$
3. $y = \cos x + \sin x \to dy = (-\sin x + \cos x)\, dx$
4. $x^2 + 4y^2 = 5 \to 2x\, dx + 8y\, dy = 0 \to dy = \frac{-x}{4y}\, dx$

Example

Find dx of each of the following functions:

1. $u = 3 - 4x^2 \to du = -8x\, dx \to dx = \frac{1}{-8x}\, du$.
2. $u = ax + b \to du = a\, dx \to dx = \frac{du}{a}$.
3. $u = 1 - \cos^2 x \to du = -2\cos x \sin x\, dx \to dx = \frac{du}{-2\cos x \sin x}$.
4. $u = \sin^{-1} x \to du = \frac{1}{\sqrt{1-x^2}}\, dx \to dx = \sqrt{1 - x^2}\, du$.

Chain Rule, Second form

Example

Let $y = \sin x + e^x$ and $x = t^2 + 4t$. Find $\frac{dy}{dt}$ at $t = 1$?

Solution: Since $t = 1$, we have $x = (1)^2 + 4(1) = 5$.

$$\begin{aligned}
\frac{dy}{dt} &= \frac{(\cos x + e^x)dx}{dt} \\
&= \frac{(\cos x + e^x)(2t+4)dt}{dt} \\
&= (\cos x + e^x)(2t+4) \\
\frac{dy}{dt}_{|t=1} &= (\cos 5 + e^5)(2(1)+4)
\end{aligned}$$

Chain Rule, Second form

Example

Let $r = \frac{2}{q} + 10q$ and $q = 7 + \frac{12}{t}t^2$. Find $\frac{dr}{dt}$ at $t = 3$?

Solution: Since $t = 3$, we have $q = 11$.

$$\frac{dr}{dt} = \frac{(-\frac{2}{q^2} + 10)dq}{dt}$$

$$= \frac{(-\frac{2}{q^2} + 10)(-\frac{12}{t^2})dt}{dt}$$

$$= \left(-\frac{2}{q^2} + 10\right)\left(-\frac{12}{t^2}\right)$$

$$\frac{dy}{dt}_{|t=3} =$$

Netwon's Method

Recall:

Theorem

(Intermediate Value Theorem) Let f be a continuous function on an interval $[a, b]$ such that $f(a)$ and $f(b)$ have different signs, then f must have a root in $[a, b]$, i.e., there exists $c \in [a, b]$ such that $f(c) = 0$.

Recall that the intermediate value theorem tells us that there will be a root, but does not tell us how to find it. We will use Newton's method to approximate the root.

1. Guess an initial approximation x_0 to the root.
2. Determine a new approximation using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example

Consider the equation $x^2 - 2 = 0$.

1. Use the intermediate value theorem to show that the equation above has a root in $[1, 2]$.
2. Use the Newton's method to approximate a root of the equation above in $[1, 2]$ with $x_0 = 1$.

Solution: (1) Let $f(x) = x^2 - 2$ (continuous) and we have $f(1) = -1 < 0$ while $f(2) = 2 > 0$, therefore by the intermediate value theorem, there must be a root in $[1, 2]$.
(2) Let $x_0 = 1$, we have $f'(x) = 2x$ and

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(1)^2 - 2}{2(1)} = 1.5$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{(1.5)^2 - 2}{2(1.5)} =$$

$$x_3 = x_3 - \frac{f(x_2)}{f'(x_2)} = - \frac{()^2 - 2}{2()} =$$