Section 3.9
Inverse Trigonometric Functions
1 Lecture

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MATHS 101: Calculus I

Inverse Trigonometric functions

1. The derivative of the basic inverse trigonometric functions.
2. The derivative of functions that involve inverse trigonometric functions.
3. Identities of inverse trigonometric functions using differential calculus.

$f(x) = \sin x$ has an *inverse* if $x \in [\frac{-\pi}{2}, \frac{\pi}{2}]$. The inverse of the sine function is denoted by

$$f^{-1}(x) = \sin^{-1} x \qquad (= \arcsin x)$$

with the following properties:

1. The graph of $y = \sin^{-1} x$ is:
2. The domain of $\sin^{-1} x$ is $[-1, 1]$
3. The range of $\sin^{-1} x$ is $[\frac{-\pi}{2}, \frac{\pi}{2}]$
4. $\sin\left(\sin^{-1} x\right) = x$
5. $\sin^{-1}(\sin x) = x$

Exercise

Write a similar definition for the $\tan^{-1} x$ and $\sec^{-1} x$ and explore their properties.

Derivative of $\sin^{-1} x$

Example

Find $\frac{d}{dx}\left(\sin^{-1} x\right)$.

Solution:

$$\begin{aligned} y &= \sin^{-1} x, \ \text{find } y' \\ \sin y &= \sin\left(\sin^{-1} x\right) \\ \sin y &= x \\ \cos y \cdot y' &= 1 \\ y' &= \frac{1}{\cos y} = \sec y \end{aligned}$$

$$\begin{aligned} \sin y = x &= \frac{x}{1}\frac{\text{opp}}{\text{hypo}} \\ (\text{adj})^2 + (\text{opp})^2 &= (\text{hypo})^2 \\ (\text{adj})^2 + x^2 &= 1 \\ (\text{adj})^2 &= 1 - x^2 \\ \text{adj} &= \sqrt{1-x^2} \\ \sec y = \frac{\text{hypo}}{\text{adj}} &= \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

$$\boxed{y' = \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}}$$

Exercise

Derive a formula for $\frac{d}{dx}\left(\cos^{-1} x\right)$

Derivative of $\tan^{-1} x$

Example

Find $\frac{d}{dx}\left(\tan^{-1} x\right)$.

Solution:

$$\begin{aligned} y &= \tan^{-1} x, \text{ find } y' \\ \tan y &= \tan\left(\tan^{-1} x\right) \\ \tan y &= x \\ \sec^2 y \cdot y' &= 1 \\ y' &= \frac{1}{\sec^2 y} = \cos^2 y \end{aligned}$$

$$\begin{aligned} \tan y = x &= \frac{x}{1}\frac{\text{opp}}{\text{adj}} \\ (\text{adj})^2 + (\text{opp})^2 &= (\text{hypo})^2 \\ 1 + x^2 &= (\text{hypo})^2 \\ (\text{hypo})^2 &= 1 + x^2 \\ \text{hyp} &= \sqrt{1+x^2} \\ \cos y = \frac{\text{adj}}{\text{hypo}} &= \frac{1}{\sqrt{1+x^2}} \end{aligned}$$

$$\boxed{y' = \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}}$$

Exercise

Derive a formula for $\frac{d}{dx}\left(\cot^{-1} x\right)$

Derivative of $\sec^{-1} x$

Example

Find $\frac{d}{dx}\left(\sec^{-1} x\right)$.

Solution:

$$y = \sec^{-1} x, \ \text{find } y'$$
$$\tan y = \sec\left(\sec^{-1} x\right)$$
$$\sec y = x$$
$$\sec y \tan y \cdot y' = 1$$
$$y' = \frac{1}{\sec y \tan y} = \frac{\cot y}{x}$$

$$\sec y = x = \frac{x}{1}\frac{\text{hypo}}{\text{adj}}$$
$$(\text{adj})^2 + (\text{opp})^2 = (\text{hypo})^2$$
$$1 + (\text{opp})^2 = x^2$$
$$(\text{opp})^2 = x^2 - 1$$
$$\text{opp} = \sqrt{x^2 - 1}$$
$$\cot y = \frac{\text{adj}}{\text{opp}} = \frac{1}{\sqrt{x^2 - 1}}$$

$$\boxed{y' = \frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2 - 1}}}$$

Exercise

Derive a formula for $\frac{d}{dx}\left(\csc^{-1} x\right)$

Summary

(Derivative of the Inverse Trigonometic functions)

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}$$

$$(\tan^{-1} x)' = \frac{1}{1+x^2} \qquad (\cot^{-1} x)' = \frac{-1}{1+x^2}$$

$$(\sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2-1}} \qquad (\csc^{-1} x)' = \frac{-1}{|x|\sqrt{x^2-1}}$$

2 - Derivative of functions that involve the inverse trigonometric functions

Example

Find y' if $y = \tan^{-1}(\sqrt{x})$.

Solution: We have $y = \tan^{-1}(\sqrt{x})$

$$y' = \frac{1}{1+(\sqrt{x})^2} \cdot \left(\sqrt{x}\right)'$$
$$= \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}}$$

Example

Find y' if $y = \sqrt{\tan^{-1} x}$.

Solution: We have $y = \sqrt{\tan^{-1} x}$

$$y' = \frac{1}{2\sqrt{\tan^{-1} x}} \cdot \left(\tan^{-1} x\right)'$$
$$= \frac{1}{2\sqrt{\tan^{-1} x}} \cdot \frac{1}{1+x^2}$$

Example

Find y' if $y = \cos^{-1}(e^x)$.

Solution: We have $y = \cos^{-1}(e^x)$

$$y' = \frac{-1}{\sqrt{1-(e^x)^2}} \cdot (e^x)'$$

$$= \frac{-1}{\sqrt{1-e^{2x}}} \cdot e^x$$

Example

Find y' if $y = \arcsin(3-2x)$.

Solution: We have $y = \arcsin(3-2x)$

$$y' = \frac{1}{\sqrt{1-(3-2x)^2}} \cdot (3-2x)'$$

$$= \frac{1}{\sqrt{1-(3-2x)^2}} \cdot (-2)$$

Example

Find y' if $y = \sin^{-1}\left(\sqrt{\sin x}\right)$.

Solution: We have $y = \sin^{-1}\left(\sqrt{\sin x}\right)$

$$y' = \frac{1}{\sqrt{1-\left(\sqrt{\sin x}\right)^2}} \cdot \left(\sqrt{\sin x}\right)'$$

$$= \frac{1}{\sqrt{1-(\sqrt{\sin x})^2}} \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x$$

$$= \frac{1}{\sqrt{1-\sin x}} \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x$$

Example

Find y' if $y = x\sin^{-1}x + \sqrt{1-x^2}$ and simplify your answer.

Solution:

$$y' = \sin^{-1}x + x \cdot \frac{1}{2\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \cdot (-2x)$$

$$y' = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$$

$$= \sin^{-1}x$$

Identities of inverse trigonometric functions using differential calculus.

Example

Find y' if $y = \sin^{-1} + \cos^{-1} x$.

Solution:

$$y' = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}$$
$$y' = 0$$

The derivative is always zero, which means the function y is a constant.

$$y = \sin^{-1} + \cos^{-1} x = C \to \sin^{-1}(0) + \cos^{-1}(0) = C \to \frac{\pi}{2} \quad = C$$

Hence the identity is

$$\boxed{\sin^{-1} + \cos^{-1} x = \frac{\pi}{2}}$$

Exercise

Show using calculus that

1. $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$
2. $\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$

Example

Find y' if $y = \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$.

Solution:

$$y' = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \frac{-1}{x^2} \to \frac{1}{1+x^2} + \frac{x^2}{1+x^2} \cdot \frac{-1}{x^2}$$

$$y' = \frac{1}{1+x^2} - \frac{1}{1+x^2} \to y' = 0$$

The derivative is always zero, which means the function y is a constant.

$$y = \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) = C \to \tan^{-1}(1) + \tan^{-1}\left(\frac{1}{1}\right) = C \to \frac{\pi}{2} \quad = C$$

Hence the identity is

$$\boxed{\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}}$$

Exercise

Show that $\sec^{-1}(-x) + \sec^{-1}(x) = \pi$.

Exercise

Show that

$$\tan^{-1} x + \tan^{-1} a = \tan^{-1}\left(\frac{x+a}{1-ax}\right)$$

and use it to compute $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3$.

Exercise

(Challenging problem) What is wrong in

$$\frac{d}{dx}\left(\sin^{-1} x + \sec^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} + \frac{1}{|x|\sqrt{x^2-1}}$$