Section 4.1 Relative Extrema 3 Lectures

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MATHS 101: Calculus I

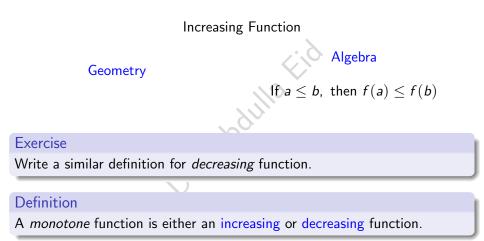
Application of Differentiation

One of the most important applications of differential calculus are the *optimization problems*, i.e., finding the optimal (best) way to do something. In our case, these optimization problem are reduced to find the minimum or maximum of a function.

Example

- Find the length that maximizes the area.
- Ind the radius that minimize the perimeter of certain circle.

1 - Monotone Functions



Question: How to tell when a function is increasing or decreasing? Answer: One way is to use the definition above, which is hard to do in general. The other way is to use Calculus as follows:

- If $f'(x) \ge 0$, then f(x) is increasing.
- If $f'(x) \leq 0$, then f(x) is decreasing.

2 - Absolute Extrema

Absolute Maximum (Global Maximum) Algebra

Geometry

f(c) is an absolute maximum (global maximum) if

 $\int f(x) \leq f(c)$, for all x

- f(c) is the absolute maximum (only one).
- c is called absolute maximizer

Exercise

Write a similar definition for *absolute minimum*.

Definition

An *absolute extrema* is either an absolute maximum or absolute minimum function.

3 - Relative Extrema

Relative Maximum (Local Maximum)

Algebra

Geometry

f(c) is an local maximum (relative maximum) if

 $f(x) \leq f(c)$, for some value of x near

- f(c) is the local maximum (maybe more than one).
- *c* is called local maximizer

Exercise

Write a similar definition for *local minimum*.

Definition

An *local extrema* (*relative extrema* is either an local maximum or loca minimum function.

- Let $f(x) = \sin x$, then
 - It has a global minimum at $\left(\frac{-\pi}{2}, -1\right), \left(\frac{3\pi}{2}, -1\right), \dots, \left(\frac{3\pi}{2}+2n\pi, -1\right)$.
 - It has a global maximum at $(\frac{-3\pi}{2}, 1), (\frac{\pi}{2}, 1), \ldots, (\frac{\pi}{2} + 2n\pi, 1).$

Example

Let $f(x) = x^2$, then

- It has a global minimum at (0,0).
- It has no global maximum.

Example

Let $f(x) = e^x$, then

- It has no global minimum.
- It has no global maximum.

Critical Points

Question: How to find the extrema (local min, local max, absolute min, absolute max)?

Answer: The following are the candidates for the extrma.

Definition

A number c is called a critical point of f(x) if either

f'(c) = 0 or f'(c) does not exist

Note: These critical points are the candidates for local maximum or local minimum.

To find these points, write the derivative as rational function, i.e.,

$$f'(x) = \frac{\text{numerator}}{\text{denominator}} \text{ and then we have}$$

$$f'(x) = 0 \rightarrow \text{numerator} = 0.$$

$$f'(x) = \text{does not exist} \rightarrow \text{denominator} = 0.$$

Find the critical points of the following function

$$f(x) = x^3 + x^2 - x$$

Solution:

We find the derivative first which is

$$f'(x) = 3x^2 + 2x - 1$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$

numerator = 0
$$3x^{2} + 2x - 1 = 0$$

$$x = -1 \text{ or } x = \frac{1}{3}$$

f'(x) does not exist denominator = 0 1 = 0Always False No Solution

Find the critical points of the following function

$$f(x) = \sqrt{3}\sin x + \cos x$$

Solution:

We find the derivative first which is

$$f'(x) = \sqrt{3}\cos x - \sin x$$

We find where the derivative equal to zero or does not exist.

f'(x) = 0numerator = 0 $\sqrt{3}\cos x - \sin x = 0$ $\sqrt{3}\cos x = \sin x$ $\sqrt{3} = \tan x \rightarrow x = \tan^{-1}\sqrt{3}$ $x = \frac{\pi}{3} + 2n\pi$ or $x = \frac{4\pi}{3} + 2n\pi$

f'(x) does not exist denominator = 0 1 = 0Always False No Solution

Find the critical points of the following function

$$f(x) = \sqrt{1 - x^2}$$

Solution:

We find the derivative first which is

$$f'(x) = \frac{-2x}{2\sqrt{1-x^2}}$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$

$$f'(x) \text{ does not exist}$$
numerator = 0
$$-2x = 0$$

$$x = 0$$

$$f'(x) \text{ does not exist}$$

Find the critical points of the following function

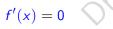
$$f(x) = \frac{x-1}{x^2 - x + 1}$$

Solution:

We find the derivative first which is

$$f'(x) = \frac{(x^2 - x + 1)(1) - (x - 1)(2x - 1)}{(x^2 - x + 1)^2} = \frac{-x^2 + 2x}{(x^2 - x + 1)^2}$$

To find the critical points, we find where the derivative equal to zero or does not exist.



numerator = 0

$$-x^2 + 2x = 0$$

$$x = 0 \text{ or } x = 2$$

f'(x) does not exist

denominator = 0

$$x^2 - x + 1 = 0$$

No Solution

Question: How to find the local min, local max?

Theorem

(First Derivative Test)

- If f'(x) changes from positive to negative as x increases, then f has a local maximum at a.
- If f'(x) changes from negative to positive as x increases, then f has a local minimum at a.

Find the intervals where the function is increasing/decreasing and find all local max/min.

$$f(x) = 2x^3 + 3x^2 - 36x$$

Solution:

We find the derivative first which is

$$f'(x) = 6x^2 + 6x - 36$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$

numerator = 0
$$6x^{2} + 6x - 36 = 0$$

$$x = 2 \text{ or } x = -3$$

f'(x) does not exist denominator = 0 1 = 0Always False No Solution

- f is increasing in (-∞, -3) ∪ (2, ∞).
 f is decreasing in (-3, 2).
- f has a local maximum at x = -3 with value f(-3) = 66. 3
- f has a local minimum at x = 2 with value f(2) = -44.

Find the intervals where the function is increasing/decreasing and find all local max/min.

$$f(x) = \sin x + \cos x, \qquad x \in [0, 2\pi]$$

Solution:

We find the derivative first which is

$$f'(x) = \cos x - \sin x$$

f'(x) = 0

numerator = 0

 $\sin x - \cos x = 0$

 $\sin x = \cos x$

$$\tan x = 1 \rightarrow x = \tan^{-1} 1$$
$$x = \frac{\pi}{4} \text{ or } x = \frac{5\pi}{4}$$

f'(x) does not exist denominator = 0 1 = 0Always False

No Solution

- *f* is increasing in $(0, \frac{\pi}{4}) \cup (\frac{5\pi}{4}, 2\pi)$.
- 2 *f* is decreasing in $(\frac{\pi}{4}, \frac{5\pi}{4})$.
- § f has a local maximum at $x = \frac{\pi}{4}$ with value $f(\frac{\pi}{4}) = \sqrt{2}$.
- f has a local minimum at $x = \frac{5\pi}{4}$ with value $f(\frac{5\pi}{4}) = -\sqrt{2}$.

(Old Exam Question) Find the intervals where the function is increasing/decreasing, find all local max/min, and sketch the graph of the function.

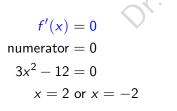
$$f(x) = x^3 - 12x + 3$$

Solution:

We find the derivative first which is

$$f'(x) = 3x^2 - 12$$

To find the critical points, we find where the derivative equal to zero or does not exist.



f'(x) does not exist denominator = 0 1 = 0Always False No Solution

Extrema

- f is increasing in $(-\infty, -2) \cup (2, \infty)$.
- f is decreasing in (-2, 2).
- § f has a local maximum at x = -2 with value f(-2) = 19.
- f has a local minimum at x = 2 with value f(2) = -13.

Exercise

Find the intervals where the function is increasing/decreasing and find all local max/min.

$$f(x) = \frac{x^2}{x - 1}$$

Solution:

We find the derivative first which is

$$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$

numerator = 0

$$x^2 - 2x = 0$$
$$x = 0 \text{ or } x = 2$$

f'(x) does not exist

denominator = 0

$$(x-1)^2 = 0$$
$$x = 1$$

- f is increasing in $(-\infty, 0) \cup (2, \infty)$.
- f is decreasing in (0, 2).
- § f has a local maximum at x = 0 with value f(0) = 0.
- f has a local minimum at x = 2 with value f(2) = 4.

Exercise

Find the intervals where the function is increasing/decreasing and find all local max/min.

$$f(x) = \sin^{-1} x$$

Solution:

We find the derivative first which is

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$f'(x) \text{ does not exist}$$
numerator = 0
$$1 = 0$$
Always False
No Solution
$$f'(x) = 0$$
denominator = 0
$$\sqrt{1 - x^2} = 0$$

$$x = -1 \text{ or } x = 1$$

Recall the domain of $f(x) = \sin^{-1} x$ which is [-1, 1].

- f is increasing in (-1, 1).
- I has no local maximum nor local minimum.

Exercise

If f is an increasing function. Show that f^{-1} is an increasing function.

Since f is an increasing function, then f' > 0. Now we find the derivative of f^{-1} which is of f^{-1} which is

$$(f^{-1}(y))' = \frac{1}{f'(f^{-1}(y))} = > 0$$