Section 4.1
Relative Extrema
3 Lectures

Dr. Abdulla Eid

College of Science

MATHS 101: Calculus I

Application of Differentiation

One of the most important applications of differential calculus are the *optimization problems*, i.e., finding the optimal (best) way to do something.In our case, these optimization problem are reduced to find the minimum or maximum of a function.

Example

1. Find the length that maximizes the area.
2. Find the radius that minimize the perimeter of certain circle.

1 - Monotone Functions

Increasing Function

Geometry

Algebra

If $a \leq b$, then $f(a) \leq f(b)$

Exercise

Write a similar definition for *decreasing* function.

Definition

A *monotone* function is either an increasing or decreasing function.

Question: How to tell when a function is increasing or decreasing?
Answer: One way is to use the definition above, which is hard to do in general. The other way is to use Calculus as follows:

- If $f'(x) \geq 0$, then $f(x)$ is increasing.
- If $f'(x) \leq 0$, then $f(x)$ is decreasing.

2 - Absolute Extrema

Absolute Maximum (Global Maximum)

Geometry

Algebra

$f(c)$ is an *absolute maximum* (*global maximum*) if

$$f(x) \leq f(c), \text{ for all } x$$

- $f(c)$ is the absolute maximum (only one).
- c is called absolute maximizer

Exercise

Write a similar definition for *absolute minimum*.

Definition

An *absolute extrema* is either an absolute maximum or absolute minimum function.

3 - Relative Extrema

Relative Maximum (Local Maximum)

Algebra

Geometry

$f(c)$ is an *local maximum* (*relative maximum*) if

$f(x) \leq f(c)$, for some value of x nea

- $f(c)$ is the local maximum (maybe more than one).
- c is called local maximizer

Exercise

Write a similar definition for *local minimum*.

Definition

An *local extrema* (*relative extrema* is either an local maximum or loca minimum function.

Example

Let $f(x) = \sin x$, then

- It has a global minimum at $(\frac{-\pi}{2}, -1), (\frac{3\pi}{2}, -1), \ldots, (\frac{3\pi}{2} + 2n\pi, -1)$.
- It has a global maximum at $(\frac{-3\pi}{2}, 1), (\frac{\pi}{2}, 1), \ldots, (\frac{\pi}{2} + 2n\pi, 1)$.

Example

Let $f(x) = x^2$, then

- It has a global minimum at (0,0).
- It has no global maximum.

Example

Let $f(x) = e^x$, then

- It has no global minimum.
- It has no global maximum.

Critical Points

Question: How to find the extrema (local min, local max, absolute min, absolute max)?
Answer: The following are the candidates for the extrma.

Definition

A number c is called a critical point of $f(x)$ if either

$$f'(c) = 0 \text{ or } f'(c) \text{ does not exist}$$

Note: These critical points are the candidates for local maximum or local minimum.
To find these points, write the derivative as rational function, i.e., $f'(x) = \frac{\text{numerator}}{\text{denominator}}$ and then we have

1. $f'(x) = 0 \rightarrow \text{numerator} = 0.$
2. $f'(x) = \text{ does not exist} \rightarrow \text{denominator} = 0.$

Example

Find the critical points of the following function

$$f(x) = x^3 + x^2 - x$$

Solution:

We find the derivative first which is

$$f'(x) = 3x^2 + 2x - 1$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$

$$\text{numerator} = 0$$

$$3x^2 + 2x - 1 = 0$$

$$x = -1 \text{ or } x = \frac{1}{3}$$

$f'(x)$ does not exist

$$\text{denominator} = 0$$

$$1 = 0$$

Always False

No Solution

Example

Find the critical points of the following function

$$f(x) = \sqrt{3}\sin x + \cos x$$

Solution:
We find the derivative first which is

$$f'(x) = \sqrt{3}\cos x - \sin x$$

We find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$
$$\text{numerator} = 0$$
$$\sqrt{3}\cos x - \sin x = 0$$
$$\sqrt{3}\cos x = \sin x$$
$$\sqrt{3} = \tan x \rightarrow x = \tan^{-1}\sqrt{3}$$
$$x = \frac{\pi}{3} + 2n\pi \text{ or } x = \frac{4\pi}{3} + 2n\pi$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Example

Find the critical points of the following function

$$f(x) = \sqrt{1 - x^2}$$

Solution:
We find the derivative first which is

$$f'(x) = \frac{-2x}{2\sqrt{1 - x^2}}$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$\begin{aligned} f'(x) &= 0 \\ \text{numerator} &= 0 \\ -2x &= 0 \\ x &= 0 \end{aligned}$$

$$\begin{aligned} f'(x) &\text{ does not exist} \\ \text{denominator} &= 0 \\ 1 - x^2 &= 0 \\ x = 1 &\text{ or } x = -1 \end{aligned}$$

Example

Find the critical points of the following function

$$f(x) = \frac{x-1}{x^2-x+1}$$

Solution:
We find the derivative first which is

$$f'(x) = \frac{(x^2-x+1)(1)-(x-1)(2x-1)}{(x^2-x+1)^2} = \frac{-x^2+2x}{(x^2-x+1)^2}$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$f'(x) = 0$	$f'(x)$ does not exist
numerator $= 0$	denominator $= 0$
$-x^2+2x = 0$	$x^2-x+1 = 0$
$x = 0$ or $x = 2$	No Solution

First Derivative Test

Question: How to find the local min, local max?

Theorem

(First Derivative Test)

1. *If $f'(x)$ changes from positive to negative as x increases, then f has a local maximum at a.*
2. *If $f'(x)$ changes from negative to positive as x increases, then f has a local minimum at a.*

Example

Find the intervals where the function is increasing/decreasing and find all local max/min.

$$f(x) = 2x^3 + 3x^2 - 36x$$

Solution:

We find the derivative first which is

$$f'(x) = 6x^2 + 6x - 36$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$

$$\text{numerator} = 0$$

$$6x^2 + 6x - 36 = 0$$

$$x = 2 \text{ or } x = -3$$

$f'(x)$ does not exist

$$\text{denominator} = 0$$

$$1 = 0$$

Always False

No Solution

Number Line

1. f is increasing in $(-\infty, -3) \cup (2, \infty)$.
2. f is decreasing in $(-3, 2)$.
3. f has a local maximum at $x = -3$ with value $f(-3) = 66$.
4. f has a local minimum at $x = 2$ with value $f(2) = -44$.

Example

Find the intervals where the function is increasing/decreasing and find all local max/min.

$$f(x) = \sin x + \cos x, \qquad x \in [0, 2\pi]$$

Solution:

We find the derivative first which is

$$f'(x) = \cos x - \sin x$$

$$\begin{aligned} f'(x) &= 0 \\ \text{numerator} &= 0 \\ \sin x - \cos x &= 0 \\ \sin x &= \cos x \\ \tan x = 1 &\rightarrow x = \tan^{-1} 1 \\ x = \frac{\pi}{4} &\text{ or } x = \frac{5\pi}{4} \end{aligned}$$

$$\begin{aligned} f'(x) &\text{ does not exist} \\ \text{denominator} &= 0 \\ 1 &= 0 \\ &\text{Always False} \\ &\text{No Solution} \end{aligned}$$

Number Line

1. f is increasing in $(0, \frac{\pi}{4}) \cup (\frac{5\pi}{4}, 2\pi)$.
2. f is decreasing in $(\frac{\pi}{4}, \frac{5\pi}{4})$.
3. f has a local maximum at $x = \frac{\pi}{4}$ with value $f(\frac{\pi}{4}) = \sqrt{2}$.
4. f has a local minimum at $x = \frac{5\pi}{4}$ with value $f(\frac{5\pi}{4}) = -\sqrt{2}$.

Example

(Old Exam Question) Find the intervals where the function is increasing/decreasing, find all local max/min, and sketch the graph of the function.

$$f(x) = x^3 - 12x + 3$$

Solution:
We find the derivative first which is

$$f'(x) = 3x^2 - 12$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$
$$\text{numerator} = 0$$
$$3x^2 - 12 = 0$$
$$x = 2 \text{ or } x = -2$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is increasing in $(-\infty, -2) \cup (2, \infty)$.
2. f is decreasing in $(-2, 2)$.
3. f has a local maximum at $x = -2$ with value $f(-2) = 19$.
4. f has a local minimum at $x = 2$ with value $f(2) = -13$.

Exercise

Find the intervals where the function is increasing/decreasing and find all local max/min.

$$f(x) = \frac{x^2}{x-1}$$

Solution:

We find the derivative first which is

$$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2}$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$
$$\text{numerator} = 0$$
$$x^2 - 2x = 0$$
$$x = 0 \text{ or } x = 2$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$(x-1)^2 = 0$$
$$x = 1$$

Number Line

1. f is increasing in $(-\infty, 0) \cup (2, \infty)$.
2. f is decreasing in $(0, 2)$.
3. f has a local maximum at $x = 0$ with value $f(0) = 0$.
4. f has a local minimum at $x = 2$ with value $f(2) = 4$.

Exercise

Find the intervals where the function is increasing/decreasing and find all local max/min.

$$f(x) = \sin^{-1} x$$

Solution:

We find the derivative first which is

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$f'(x)$ does not exist

$$\text{denominator} = 0$$

$$\sqrt{1-x^2} = 0$$

$$x = -1 \text{ or } x = 1$$

$f'(x) = 0$

$$\text{numerator} = 0$$

$$1 = 0$$

Always False

No Solution

Number Line

Recall the domain of $f(x) = \sin^{-1} x$ which is $[-1, 1]$.

1. f is increasing in $(-1, 1)$.
2. f has no local maximum nor local minimum.

Exercise

If f is an increasing function. Show that f^{-1} is an increasing function.

Solution:

Since f is an increasing function, then $f' > 0$. Now we find the derivative of f^{-1} which is

$$\left(f^{-1}(y)\right)' = \frac{1}{f'(f^{-1}(y))} = \; > 0$$