Section 4.3
Concavity and Curve Sketching
1.5 Lectures

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MATHS 101: Calculus I

Concavity

Increasing Function has three cases

Question: How to distinguish between these three types of behavior?
Answer: Recall: If $g(x)$ is increasing, then $g'(x) > 0$.

1. If $f''(x) > 0$, then the curve is concave upward (CU).
2. If $f''(x) < 0$, then the curve is concave downward (CD).
3. If $f''(x) = 0$ (for all x), then $f(x)$ has no curvature (line).

Inflection Points

Definition

A number c is called an inflection point of $f(x)$ if at these point, the function changes from concave upward to downward and vice verse. The candidates are the points c, where

$$f''(c) = 0 \text{ or } f''(c) \text{ does not exist}$$

Example

Discuss the following curve with respect to concavity and inflection points.

$$f(x) = x^3$$

Solution:

We find the derivatives first which are

$$f'(x) = 3x^2$$
$$f''(x) = 6x$$

To find the inflection points, we find where the second derivative equal to zero or does not exist.

$$f''(x) = 0$$
$$\text{numerator} = 0$$
$$6x = 0$$
$$x = 0$$

$f'(x)$ does not exist

$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is CD in $(-\infty, 0)$.
2. f is CU in $(0, \infty)$.
3. f has inflection point at $x = 0$ with value $f(0) = 0$.

Example

Discuss the following curve with respect to concavity and inflection points.

$$f(x) = 2 + \sin x, \qquad x \in [0, 2\pi]$$

Solution:

We find the derivatives first which are

$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$

To find the inflection points, we find where the second derivative equal to zero or does not exist.

$$f'(x)0$$
$$\text{numerator} = 0$$
$$\sin x = 0$$
$$x = 0 \text{ or } x = \pi$$

$$f''(x) = \text{ does not exist}$$
$$\text{denominator} = 0$$
$$-1 = 0$$

Always False

No Solution

Number Line

1. f is CD in $(\pi, 2\pi)$.
2. f is CD in $(0, \pi)$.
3. f has an inflection point at $(\pi, 0)$.

Example

(Old Exam Question) Discuss the following curve with respect to concavity and inflection points.

$$f(x) = x^4 - 3x^3 + 3x^2 - 5$$

Solution:

We find the derivatives first which are

$$f'(x) = 4x^3 - 9x^2 + 6x$$
$$f''(x) = 12x^2 - 18x + 6$$

$$f''(x) = 0$$
$$\text{numerator} = 0$$
$$12x^2 - 18x + 6 = 0$$
$$x = 1 \text{ or } x = \frac{1}{2}$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is CD in $(\frac{1}{2}, 1)$.
2. f is CU in $(-\infty, \frac{1}{2}) \cup (1, \infty)$.
3. f has inflection point at $x = \frac{1}{2}$ with value $f(\frac{1}{2}) =$ and at $x = 1$ with value $f(1) =$.

Exercise

(All in All) Find the intervals where the function is increasing/decreasing, concave upward, concave downward, find all local max/min, find inflection points and sketch the graph of the function.

$$f(x) = x^3 - 6x^2 + 9x + 1$$

Solution: We find the derivative first which is

$$f'(x) = 3x^2 - 12x + 9$$
$$f''(x) = 6x - 12$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$
$$\text{numerator} = 0$$
$$3x^2 - 12x + 9 = 0$$
$$x = 1 \text{ or } x = 3$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is increasing in $(-\infty, 1) \cup (3, \infty)$.
2. f is decreasing in $(1, 3)$.
3. f has a local maximum at $x = 1$ with value $f(1) = 5$.
4. f has a local minimum at $x = 3$ with value $f(3) = 1$.

Recall that the derivatives are

$$f'(x) = 3x^2 - 12x + 9$$
$$f''(x) = 6x - 12$$

To find the inflection points, we find where the second derivative equal to zero or does not exist.

$$\begin{aligned} f''(x) &= 0 \\ \text{numerator} &= 0 \\ 6x - 12 &= 0 \\ x &= 2 \end{aligned}$$

$f'(x)$ does not exist

$$\begin{aligned} \text{denominator} &= 0 \\ 1 &= 0 \end{aligned}$$

Always False

No Solution

Number Line

1. f is CD in $(-\infty, 2)$.
2. f is CU in $(2, \infty)$.
3. f has inflection point at $x = 2$ with value $f(2) = 3$.

General Guidelines to sketch the curve of a rational function $y = f(x)$

1. Find the domain of $f(x)$.
2. Find the x–intercept by solving $y = 0$ (whenever possible). Find the y–intercept by setting $x = 0$ to find $y = f(0)$ (if possible).
3. Find the horizontal and vertical asymptotes.
4. Find the local maximum, local minimum, inflection points the interval where the function is increasing, decreasing, concave upward, concave downward.

Example

(All in All) Find the intervals where the function is increasing/decreasing, concave upward, concave downward, find all local max/min, find inflection points and sketch the graph of the function.

$$f(x) = x - \sin x, \qquad x \in [0, 4\pi]$$

Solution: We find the derivative first which is

$$f'(x) = 1 - \cos x$$
$$f''(x) = -\sin x$$

$f'(x) = 0$

numerator $= 0$

$1 - \cos x = 0 \rightarrow \cos x = 1$

$x = 0$ or $x = 2\pi$ or $x = 4\pi$

$f'(x)$ does not exist

denominator $= 0$

$1 = 0$

Always False

No Solution

Number Line

1. f is increasing in $(0, 2\pi) \cup (2\pi, 4\pi)$.
2. f is not decreasing on $[0, 4\pi]$.
3. f has no local maximum nor local minimum.

Recall that the derivatives are

$$f'(x) = 1 - \cos x$$
$$f''(x) = \sin x$$

$f''(x) = 0$

$$\text{numerator} = 0$$
$$\sin x = 0$$
$$x = 0 \text{ or } x = \pi \text{ or } x = 2\pi$$
$$\text{or } x = 2\pi \text{ or } 4\pi$$

$f'(x)$ does not exist

$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is CD in $(0, \pi) \cup (2\pi, 3\pi)$.
2. f is CU in $(\pi, 2\pi) \cup (3\pi, 4\pi)$.
3. f has inflection point at $x = \pi, 2\pi, 3\pi$ with points (π, π), $(2\pi, 2\pi)$, and $(3\pi, 3\pi)$.

Exercise

(All in All) Find the intervals where the function is increasing/decreasing, concave upward, concave downward, find all local max/min, find inflection points and sketch the graph of the function.

$$f(x) = x^5 - 4x^4$$

Solution: We find the derivative first which is

$$f'(x) = 5x^4 - 20x^3$$
$$f''(x) = 20x^3 - 60x^2$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$
$$\text{numerator} = 0$$
$$5x^4 - 20x^3 = 0$$
$$x = 0 \text{ or } x = 4$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is increasing in $(-\infty, 0) \cup (4, \infty)$.
2. f is decreasing in $(0, 4)$.
3. f has a local maximum at $x = 0$ with value $f(0) = 0$.
4. f has a local minimum at $x = 4$ with value $f(4) =$.

Recall that the derivatives are

$$f'(x) = 5x^4 - 20x^3$$
$$f''(x) = 20x^3 - 60x^2$$

To find the inflection points, we find where the second derivative equal to zero or does not exist.

$$f''(x) = 0$$
$$\text{numerator} = 0$$
$$20x^3 - 60x^2 = 0$$
$$x = 1 \text{ or } x = 3$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is CD in $(-\infty, 3)$.
2. f is CU in $(3, \infty)$.
3. f has inflection point at $x = 3$ with value $f(3) =$.

Example

Sketch the function

$$y = f(x) = \frac{x}{x-2}$$

Solution: 1- The domain of $f(x)$ is all real numbers except $x = 2$.
2- x–intercept: solve

$$y = 0 \to \frac{x}{x-2} = 0 \to x = 0$$

so $(0,0)$ is the x–intercept. Moreover, it is the y–intercept.
3- Horizontal Asympotote:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x-2} = 1$$

So $y = 1$ is the horizontal asymptote. Moreover, $x = 2$ is the vertical asymptote.

4- Local max, local min

Example

Sketch the function

$$y = f(x) = \frac{x}{x-2}$$

Solution: We find the derivative first which is

$$f'(x) = \frac{-2}{(x-2)^2}$$

$$f''(x) = \frac{4}{(x-2)^3}$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$f'(x)$ does not exist

$$\text{numerator} = 0$$

$$-2 = 0$$

Always False

$f'(x) = 0$

$$\text{denominator} = 0$$

$$x - 2 = 0$$

Number Line

1. f is decreasing in $(-\infty, 2) \cup (2, \infty)$.
2. f has no local maximum or local minimum.

Recall that the derivatives are

$$f'(x) = \frac{-2}{(x-2)^2}$$
$$f''(x) = \frac{4}{(x-2)^3}$$

$f''(x)$ does not exist

$$\text{numerator} = 0$$
$$4 = 0$$

Always False

No Solution

$f''(x) = 0$

$$\text{denominator} = 0$$
$$x - 2 = 0$$
$$x = 2$$

Number Line

1. f is CD in $(-\infty, 2)$.
2. f is CU in $(2, \infty)$.
3. f has no inflection point at $x = 2$ since it is not in the domain of $f(x)$.