Section 4.6
Optimization Problem
2 Lecture

Dr. Abdulla Eid

College of Science

MATHS 101: Calculus I

Second Derivative Test

Theorem

(Second Derivative Test) Suppose $f'(c) = 0$, *then*

1. *If* $f''(c) < 0$, *then* f *has a local maximum at* c.
2. *If* $f''(c) > 0$, *then* f *has a local minimum at* c.

Notes:

- If $f''(c) = 0$, then we say the second derivative test is inconclusive! and in that case we need to use the first derivative test.
- This is very useful for this section to check quickly if we have a maximizer or minimizer.

Example

Find the local maximum and local minimum (if any) using the second derivative test.

$$f(x) = x^3 - 12x + 1$$

Solution:

We find the derivatives first which are

$$f'(x) = 3x^2 - 12$$
$$f''(x) = 6x$$

To find the critical points, we find where the first derivative equal to zero or does not exist.

$f'(x) = 0$

$$\text{numerator} = 0$$
$$3x^2 - 12 = 0$$
$$x = -2 \text{ or } x = 2$$

$f'(x)$ does not exist

$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Second Derivative Test

$$f''(-2) = 6(-2) = -12 < 0$$

So $x = -2$ is a local maximizer with maximum $f(-2) =$.

$$f''(2) = 6(2) = -2 > 0$$

So $x = 2$ is a local minimizer with minimum $f(2) =$.

Example

Find the local maximum and local minimum (if any) using the second derivative test.

$$f(x) = 7 - 2x^4$$

Solution:

We find the derivatives first which are

$$f'(x) = -8x^3$$
$$f''(x) = -24x^2$$

To find the critical points, we find where the first derivative equal to zero or does not exist.

$$f'(x) = 0$$
$$\text{numerator} = 0$$
$$-8x^3 = 0$$
$$x = 0$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Second Derivative Test

$$f''(0) = -246(0) = 0$$

So the second derivative test is inconclusive! and we apply the first derivative test.

Example

Let $f(x)$ be continuous function and have critical numbers 1 and 2. The second derivative is given by $f''(x) = 3x^2 - 16x + 17$. Where does f have a local maximum or local minimum?

Solution:

$$f''(1) = 3(1)^2 - 16(1) + 17 = 4 > 0$$

So $x = 1$ is a local minimizer.

$$f''(2) = 3(2)^2 - 16(2) + 17 = -3$$

So $x = 2$ is a local maximizer.

Example

What is the smallest perimeter possible for a rectangle whose area is 4 cm^2? What are the dimensions?

Solution:

Given: Area $= 4 \Rightarrow A = 4 = \ell w \Rightarrow \ell = \frac{4}{w}$

Required: Minimize Perimeter

$$\text{Required: Minimize } P = 2\ell + 2w$$
$$P = 2\frac{4}{w} + 2w$$
$$P = \frac{8}{w} + 2w$$

We find the derivatives first which are

$$P' = \frac{-8}{w^2} + 2 = \frac{-8 + 2w^2}{w^2}$$
$$P'' = \frac{16}{w^3}$$

Critical Points

To find the critical points, we find where the first derivative equal to zero or does not exist.

$$P'(x) = 0$$
$$\text{numerator} = 0$$
$$-8 + 2w^2 = 0$$
$$w = 2 \text{ or } w = -2 \, (\text{rejected})$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$w^2 = 0$$
$$w = 0 \, (\text{rejected})$$

Maximum or Minimum

To find out whether $w = 2$ is a maximizer or minimizer, we will use the second derivative test. so we check

$$P''(2) = \frac{16}{(2)^3} > 0$$

So $w = 2$ cm is a local minimizer with minimizer length $\ell = \frac{4}{2} = 2$ cm and minimum perimeter $P(2) = 8$ cm.

Exercise

Find the smallest area of a rectangle whose perimeter is 32 cm.

Example

Determine the dimensions of the rectangle of the largest area that can be inscribed in the right triangle with sides 3,4 , and 5?.

Solution:

Given: $\tan\theta = \frac{4}{3} = \frac{\ell}{3-w} \Rightarrow 4(3-w) = 3\ell \Rightarrow \ell = \frac{4}{3}(3-w)$

Required: Maximize Area

$$\text{Required: Maximize } A = \ell w$$

$$P = \frac{4}{2}(3-w)w$$

$$P = 4w\frac{4}{3}w^2$$

We find the derivatives first which are

$$A' = 4 - \frac{8}{3}w$$

$$P'' = \frac{-8}{3}$$

Critical Points

To find the critical points, we find where the first derivative equal to zero or does not exist.

$$
\begin{aligned}
A'(x) &= 0 \\
\text{numerator} &= 0 \\
4 - \frac{8}{3} &= 0 \\
w &= \frac{3}{2}
\end{aligned}
$$

$$
\begin{aligned}
A'(x) &\text{ does not exist} \\
\text{denominator} &= 0 \\
1 &= 0
\end{aligned}
$$

Always False

No Solution

Maximum or Minimum

To find out whether $w = \frac{3}{2}$ is a maximizer or minimizer, we will use the second derivative test. so we check

$$A''(\frac{3}{2}) = \frac{-8}{3} < 0$$

So $w = \frac{3}{2}$ cm is a local maximizer with maximizer length $\ell = \frac{4}{3}(3 - \frac{3}{2}) = 2$ cm and maximum perimeter $P(2) = 3$ cm^2.

Example

Find the point on the curve $y = f(x)$ that is closest to the point $(3, 0)$

Solution:

Given: $y = \sqrt{x}$

Required: minimum distance

$$\text{Required: minimize } D = (x-3)^2 + (y-0)^2$$

$$D = x^2 - 6x + 9 + (\sqrt{x})^2$$

$$D = x^2 - 5x + 9$$

We find the derivatives first which are

$$D' = 2x - 5$$

$$D'' = 2$$

Critical Points

To find the critical points, we find where the first derivative equal to zero or does not exist.

$$D'(x) = 0$$
$$\text{numerator} = 0$$
$$2x - 5 = 0$$
$$x = \frac{5}{2}$$

$D'(x)$ does not exist

$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Maximum or Minimum

To find out whether $x = \frac{5}{2}$ is a maximizer or minimizer, we will use the second derivative test. so we check

$$D''(\frac{5}{2}) = 2 > 0$$

So $x = \frac{5}{2}$ is a local minimizer with a minimum distance $D = 2.75$.

Example

What is the minimum vertical distance between the curves $y = x^2 + 1$ and $y = x - x^2$.

Solution:

$$\text{Required: minimum distance}$$
$$\text{Required: minimize } D = (x^2+1) - (x - x^2)$$
$$D = 2x^2 - x + 1$$

We find the derivatives first which are

$$D' = 4x - 1$$
$$D'' = 4$$

Critical Points

To find the critical points, we find where the first derivative equal to zero or does not exist.

$$D'(x) = 0$$
$$\text{numerator} = 0$$
$$4x - 1 = 0$$
$$x = \frac{1}{4}$$

$$D'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Maximum or Minimum

To find out whether $x = \frac{1}{24}$ is a maximizer or minimizer, we will use the second derivative test. so we check

$$D''(\frac{1}{4}) = 4 > 0$$

So $x = \frac{1}{4}$ is a local minimizer with a minimum distance $D = \frac{13}{16}$.

Example

Find a number for which the sum of it and its reciprocal is the smallest possible.

Solution:
Given: nothing

Required: Minimize Sum

$$\text{Required: Minimize } S = x + \frac{1}{x}$$

We find the derivatives first which are

$$S' = 1 + \frac{-1}{x^2} = \frac{x^2 - 1}{x^2}$$

$$P'' = \frac{2}{x^3}$$

Critical Points

To find the critical points, we find where the first derivative equal to zero or does not exist.

$$\begin{aligned} S'(x) &= 0 \\ \text{numerator} &= 0 \\ x^2 - 1 &= 0 \\ w = 1 \text{ or } w &= -1 \end{aligned}$$

$$\begin{aligned} S'(x) & \text{ does not exist} \\ \text{denominator} &= 0 \\ x^2 &= 0 \\ x &= 0 \,(\text{rejected}) \end{aligned}$$

Maximum or Minimum

To find out whether $x = 1$ or $x = -1$ is a maximizer or minimizer, we will use the second derivative test. so we check

$$S''(1) = \frac{2}{(1)^3} > 0$$

So $x = 1$ cm is a local minimizer with minimum sum $S(1) = 2$ cm.

Exercise

Find a positive number for which the sum of its reciprocal and four times its square is the smallest possible.

Example

For what values of a make the function

$$f(x) = x^2 + ax$$

have a local minimum at $x = 2$.

Solution:

We find the derivatives first which are

$$f'(x) = 2x + a$$

$$f \text{ has a local minimum at } x = 2 \Rightarrow f'(3) = 0 \Rightarrow 4 + a = 0$$

Solving the above equation yield that $a = -4$

Example

For what values of a and b make the function

$$f(x) = x^3 + ax^2 + bx$$

have a local maximum at $x = -1$ and local minimum at $x = 3$.

Solution:
We find the derivatives first which are

$$f'(x) = 3x^2 + 2ax + b$$

$$f \text{ has a local mamximum at } x = -1 \Rightarrow f'(-1) = 0 \Rightarrow 3 - 2a + b = 0$$

$$f \text{ has a local minimum at } x = 3 \Rightarrow f'(3) = 0 \Rightarrow 27 + 6a + b = 0$$

Solving the above two equations yield that $a = 3$ and $b = 3$

Exercise

Solve the previous example, but this time, assume the function has a local minimum at $x = 4$ and a point of inflection at $x = 1$.