

# Section 5.4

## Indefinite Integrals

### 2 Lectures

Dr. Abdulla Eid

College of Science

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# Indefinite Integral

Given a function  $f$ , if  $F$  is a function such that

$$F'(x) = f(x)$$

then  $F$  is called **antiderivative** of  $f$ .

## Definition

An antiderivative of  $f$  is simply a function whose derivative is  $f$ .

**Note:** Any two antiderivatives of a function differ only by a constant.

# Indefinite Integrals

If  $F(x)$  is the antiderivative of  $f(x)$ , we will write

$$\int f(x) dx = \underbrace{F(x)}_{\text{antiderivative}} + C$$

where

- The symbol  $\int$  is called the **integral sign**.
- The function  $f(x)$  is called the **integrand**.
- The constant  $C$  is called the **constant of integration**.
- $dx$  indicates the variable involved in the integration which is  $x$ .

**Note: The Fundamental Theorem of Calculus**

$$\frac{d}{dx} \left( \int f(x) dx \right) = f(x) \text{ and } \int \frac{d}{dx} (f(x)) dx = f(x)$$

Integration and differentiation are reversing each other.

## Examples

### Example

Find  $\int 7 dx$ .

Solution: We need to find what is the function that if we differentiate it we get 7?

$$\int 7 dx = 7x + C$$

### Example

Find  $\int x dx$ .

Solution: We need to find what is the function that if we differentiate it we get  $x$ ?

$$\int x dx = \frac{1}{2}x^2 + C$$

## Examples

### Example

Find  $\int x^9 dx$ .

Solution: We need to find what is the function that if we differentiate it we get  $x^9$ ?

$$\int x^9 dx = \frac{1}{10}x^{10} + C$$

### Example

Find  $\int \frac{1}{x} dx$ .

Solution: We need to find what is the function that if we differentiate it we get  $\frac{1}{x}$ ?

$$\int \frac{1}{x} dx = \ln x + C$$

## Elementary integration formula

$$\textcircled{1} \int k \, dx = kx + C.$$

$$\textcircled{2} \int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \quad n \neq -1.$$

$$\textcircled{3} \int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln x + C \quad x > 0.$$

$$\textcircled{4} \int e^x \, dx = e^x + C.$$

$$\textcircled{5} \int kf(x) \, dx = k \int f(x) \, dx.$$

$$\textcircled{6} \int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx.$$

### Example

Find  $\int 5x^{-9} dx$ .

Solution:

$$\int 5x^{-9} dx = \frac{5}{-8}x^{-8} + C$$

### Example

Find  $\int \frac{3}{x^6} dx$ .

Solution:

$$\int \frac{3}{x^6} dx = \int 3x^{-6} dx = \frac{3}{-5}x^{-5} + C$$

### Example

Find  $\int 4x^6 + 3x^4 + 2x + 9 + \frac{1}{x} dx$ .

Solution:

$$\int 4x^6 + 3x^4 + 2x + 9 + \frac{1}{x} dx = \frac{4}{7}x^7 + \frac{3}{5}x^5 + x^2 + 9x + \ln x + C$$

### Example

Find  $\int x^{9.9} - 7x^6 + 3x^{-4} + x^{-1} + \sqrt{2} dx$ .

Solution:

$$x^{9.9} - 7x^6 + 3x^{-4} + x^{-1} + \sqrt{2} = \frac{1}{10.9}x^{10.9} - x^7 + \frac{3}{-3}x^{-3} + \ln x + \sqrt{2}x + C$$



### Example

Find  $\int \sqrt{x} + \frac{5}{3\sqrt[3]{x^2}} dx$ .

Solution:

$$\int \sqrt{x} + \frac{5}{3\sqrt[3]{x^2}} dx = \int x^{\frac{1}{2}} + \frac{5}{3}x^{-\frac{2}{3}} dx = \frac{2}{3}x^{\frac{3}{2}} + \frac{5}{3}3x^{\frac{1}{3}} + C$$

### Example

Find  $\int e^x + x^e + e^2 dx$ .

Solution:

$$\int e^x + x^e + e^2 dx = e^x + \frac{1}{e+1}x^{e+1} + e^2x + C$$

### Example

Find  $\int x^{-2}(4x^3 + 3x + 5) dx$ .

Solution:

$$\int x^{-2}(4x^3 + 3x + 5) dx = \int 4x + 3x^{-1} + 5x^{-2} dx = 2x + 3 \ln x - 5x^{-1} + C$$

### Example

Find  $\int \frac{x^4 + 10x}{x^2} dx$ .

Solution:

$$\int \frac{x^4 + 10x}{x^2} dx = \int x^2 + 10x^{-1} dx = \frac{1}{3}x^3 + 10 \ln x + C$$

### Example

Find  $\int (x + 2)^2 dx$ .

Solution:

$$\int (x + 2)^2 dx = \int x^2 + 4x + 4 dx = \frac{1}{3}x^3 + 2x^2 + 4x + C$$

### Example

Find  $\int \frac{d}{dx} \left( \frac{1}{\sqrt{1+x^3}} \right) dx$ .

Solution:

$$\int \frac{d}{dx} \left( \frac{1}{\sqrt{1+x^3}} \right) dx = \frac{1}{\sqrt{1+x^3}} + C$$

### Example

(Old Exam Question) Find  $\int (7x^3 - 6x^2 - \ln 3) dx$ .

Solution:

$$\int (7x^3 - 6x^2 - \ln 3) dx = \frac{7}{4}x^4 - 2x^3 - (\ln 3)x + C$$

### Example

(Old Exam Question) Find  $\int \frac{-3}{x^{3.3}} dx$ .

Solution:

$$\int \frac{-3}{x^{3.3}} dx = \int -3x^{-3.3} dx = \frac{-3}{-2.3}x^{-2.3} + C$$

## Example

(Old Exam Question) Find  $\int (8 - 5e^x) dx$ .

Solution:

$$\int (8 - 5e^x) dx = 8x - 5e^x + C$$

## Example

(Old Exam Question) Find  $\int (\sqrt{x} - \frac{3}{x}) dx$ .

Solution:

$$\int (\sqrt{x} - \frac{3}{x}) dx = \int (x^{\frac{1}{2}} - \frac{3}{x}) dx = \frac{2}{3}x^{\frac{3}{2}} - 3 \ln x + C$$

### Example

Find  $\int e^{\ln(x^2+1)} dx$ .

Solution:

$$\int e^{\ln(x^2+1)} dx = \int (x^2 + 1) dx = \frac{x^3}{3} + x + C$$

### Example

Find  $\int dx$ .

Solution:

$$\int dx = \int 1 dx = x + C$$

## Indefinite Integrals involving Trigonometric and inverse trigonometric functions

### Example

Find  $\int \csc^2 x \, dx$ .

Solution:

$$\int \csc^2 x \, dx = -\cot x + C$$

### Example

Find  $\int 4 \sin x + 3 \cos x \, dx$ .

Solution:

$$\int 4 \sin x + 3 \cos x \, dx = -4 \cos x + 3 \sin x + C$$

### Example

Find  $\int \frac{x^3 - x^4 \cos x + 5}{x^4} dx$ .

Solution:

$$\begin{aligned}\int \frac{x^3 - x^4 \cos x + 5}{x^4} &= \int \frac{1}{x} - \cos x + 5x^{-4} dx \\ &= \ln|x| - \sin x - \frac{5}{2}x^{-3} + C\end{aligned}$$

### Example

Find  $\int \frac{7}{\sqrt{1-x^2}} dx$ .

Solution:

$$\int \frac{7}{\sqrt{1-x^2}} dx = 7 \sin^{-1} x + C$$



## Example

Find  $\int \frac{1+\cos^2 x}{\cos^2 x} dx$ .

Solution:

$$\begin{aligned}\int \frac{1 + \cos^2 x}{\cos^2 x} &= \int \frac{1}{\cos^2 x} + 1 dx \\ &= \int \sec^2 x + 1 dx \\ &= \tan x + x + C\end{aligned}$$

## Example

Find  $\int \frac{\sin 2x}{\sin x} dx$ .

Solution:

$$\begin{aligned}\int \frac{\sin 2x}{\sin x} dx &= \int \frac{2 \sin x \cos x}{\sin x} dx \\ &= \int 2 \cos x dx \\ &= 2 \sin x + C\end{aligned}$$

## Example

Find  $\int \frac{t^2-1}{t^4-1} dx$ .

Solution:

$$\begin{aligned}\int \frac{t^2-1}{t^4-1} &= \int \frac{(t^2-1)}{(t^2+1)(t^2-1)} dx \\ &= \int \frac{1}{t^2+1} dx \\ &= \tan^{-1} x + C\end{aligned}$$

## Example

Find  $\int \cos x(\tan x + \sec x) dx$ .

Solution:

$$\begin{aligned}\int \cos x(\tan x + \sec x) dx &= \int \cos x \tan x + \cos x \sec x dx \\ &= \int \cos x \frac{\sin x}{\cos x} + \cos x \frac{1}{\cos x} dx \\ &= \int (\sin x + 1) dx \\ &= -\cos x + x + C\end{aligned}$$

Note: If  $\int f(x) dx = F(x) + C$ , then  $\int f(kx) dx = \frac{1}{k}F(kx) + C$ .

### Example

$$\textcircled{1} \int e^{2x} dx = \frac{1}{2}e^{2x} + C.$$

$$\textcircled{2} \int e^{-x} dx = \frac{1}{-1}e^{-x} + C.$$

$$\textcircled{3} \int e^{-2x} dx = \frac{1}{-2}e^{-2x} + C.$$

## Example

(Old Exam Question) Find  $\int (e^x + 5)^2 dx$ .

Solution:

$$\begin{aligned}\int (e^x + 5)^2 dx &= \int (e^{2x} + 10e^x + 25) dx \\ &= \frac{1}{2}e^{2x} + 10e^x + 25x + C\end{aligned}$$

## Exercise

(Old Exam Question) Find  $\int (e^x - e^{-x})^2 dx$ .

## Example

(Old Exam Question) Find  $\int \frac{1+e^x}{e^x} dx$ .

Solution:

$$\begin{aligned}\int \frac{1+e^x}{e^x} dx &= \int \left( \frac{1}{e^x} + \frac{e^x}{e^x} \right) dx \\ &= \int (e^{-x} + 1) dx \\ &= -e^{-x} + x + C\end{aligned}$$

## Example

$$\textcircled{1} \int \sin(2x) dx = -\frac{1}{2} \cos(2x) + C.$$

$$\textcircled{2} \int \cos(-x) dx = \frac{1}{-1} \sin(-x) + C.$$

$$\textcircled{3} \int \sec(\pi x) \tan(\pi x) dx = \frac{1}{\pi} \sec(\pi x) + C.$$

$$\textcircled{4} \int \sec^2(7x) dx = \frac{1}{7} \tan(7x) + C.$$

Dr.



Recall: The trigonometric Identities:

(cos and sin functions)

①  $\cos^2 x + \sin^2 x = 1.$

②  $1 - \sin^2 x = \cos^2 x.$

③  $1 - \cos^2 x = \sin^2 x.$

(tan and sec functions)

①  $\sec^2 x - \tan^2 x = 1.$

②  $\sec^2 = 1 + \tan^2 x.$

③  $\tan^2 x = \sec^2 x - 1.$

(Double Angle Formula)

①  $\sin(2x) = 2 \sin x \cos x.$

②  $\cos(2x) = \cos^2 x - \sin^2 x.$

③  $\cos^2 x = \frac{1}{2} (1 + \cos(2x)).$

④  $\sin^2 x = \frac{1}{2} (1 - \cos(2x)).$

### Example

Find  $\int \sec^2 x \, dx$ .

Solution:

$$\int \sec^2 x \, dx = \tan x + C$$

### Example

Find  $\int \tan^2 x \, dx$ .

Solution:

$$\begin{aligned}\int \tan^2 x \, dx &= \int \sec^2 x - 1 \, dx \\ &= \tan x - x + C\end{aligned}$$

## Example

Find  $\int \sin^2 x \, dx$ .

Solution:

$$\begin{aligned}\int \sin^2 x \, dx &= \int \frac{1}{2} (1 - \cos(2x)) \, dx \\ &= \frac{1}{2} \int (1 - \cos(2x)) \, dx \\ &= \frac{1}{2} \left( x - \frac{1}{2} \sin(2x) \right) + C\end{aligned}$$

## Exercise

Find  $\int \cos^2 x \, dx$

### Example

Find  $\int 6 - \cot^2 x \, dx$ .

Solution:

$$\begin{aligned}\int 6 - \cot^2 x \, dx &= \int 6 - (\csc^2 x - 1) \, dx \\ &= \int 7 - \csc^2 x \, dx \\ &= 7x - \cot x + C\end{aligned}$$

## Example

Find  $\int (\sec x + \tan x)^2 dx$ .

Solution:

$$\begin{aligned}\int (\sec x + \tan x)^2 dx &= \int \sec^2 x + 2 \sec x \tan x + \tan^2 x dx \\ &= \int \sec^2 x + 2 \sec x \tan x + \sec^2 x - 1 dx \\ &= \int 2 \sec^2 x + 2 \sec x \tan x - 1 dx \\ &= 2 \tan x + 2 \sec x - x + C\end{aligned}$$

### Example

Find  $\int \frac{1}{\sqrt{4-x^2}} dx$ .

Solution:

$$\int \frac{1}{\sqrt{4-x^2}} dx = \sin^{-1}\left(\frac{x}{2}\right)$$

### Example

Find  $\int \frac{1}{9+x^2} dx$ .

Solution:

$$\int \frac{1}{9+x^2} dx = \frac{1}{3} \sin^{-1}\left(\frac{x}{3}\right)$$