Section 5.5
More Integration Formula (The Substitution Method)
2 Lectures

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MATHS 101: Calculus I

The Substitution Method

Idea: To replace a relatively complicated integral by a simpler one (one from the list). This is done by adding an extra variable which we will call it u.

Theorem 1

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du$$

Two properties we are looking for in u:

1. u should be an inner function.
2. Almost the derivative of u appears in the integral.

Example 2

Find $\int xe^{x^2}\,dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = x^2$$

$$du = 2xdx \rightarrow dx = \frac{du}{2x}$$

$$\begin{aligned}\int xe^{x^2}\,dx &= \int xe^{u}\,\frac{du}{2x}\\ &= \frac{1}{2}\int e^{u}\,du\\ &= \frac{1}{2}e^{u} + C\\ &= \frac{1}{2}e^{x^2} + C\end{aligned}$$

Exercise 3

Find $\int \sqrt{3x+5}\,dx$.

Example 4

Find $\int \frac{x^2}{\sqrt{1-x^3}}\,dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = 1 - x^3$$

$$du = -3x^2 dx \rightarrow dx = \frac{du}{-3x^2}$$

$$\int \frac{x^2}{\sqrt{1-x^3}}\,dx = \int \frac{x^2}{\sqrt{u}} \frac{du}{-3x^2} = \frac{1}{-3}\int \frac{1}{\sqrt{u}}\,du$$

$$= \frac{1}{-3}\int (u)^{-\frac{1}{2}}\,du = \frac{2}{-3}u^{\frac{1}{2}} + C$$

$$= \frac{2}{-3}(1-x^3)^{\frac{1}{2}} + C$$

Exercise 5

Find $\int \cos x\sqrt{1+\sin x}\,dx$.

Example 6

Find $\int \frac{(\ln x)^2}{x}\, dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = \ln x$$
$$du = \frac{1}{x} dx \rightarrow dx = x du$$

$$\int \frac{(\ln x)^2}{x}\, dx = \int \frac{(u)^2}{x}\, x du = \int (u)^2\, du$$
$$= \frac{1}{3} u^3 + C$$
$$= \frac{1}{3} (\ln x)^3 + C$$

Exercise 7

Find $\int \frac{\sec^2 x}{\sqrt{\tan x}}\, dx$.

Exercise 8

(Old Exam Question) Find $\int x\sqrt{3-4x^2}\,dx$.

Example 9

Find $\int \frac{1}{ax+b}\, dx (a \neq 0)$

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = ax + b$$

$$du = adx \rightarrow dx = \frac{du}{a}$$

$$\int \frac{1}{ax+b} dx = \int \frac{1}{u} \frac{du}{a} = \frac{1}{a} \int \frac{1}{u}\, du$$
$$= \frac{1}{a} \ln u + C$$
$$= \frac{1}{a} \ln(ax+b) + C$$

Example 10

Find $\int \frac{\sin(2x)}{1-\cos^2 x}\,dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = 1 - \cos^2 x$$

$$du = (-2\cos x \sin x)dx \to dx = \frac{du}{-2\cos x \sin x}$$

$$\int \frac{\sin(2x)}{1-\cos^2 x}\,dx = \int \frac{\sin(2x)}{u} \frac{du}{-2\cos x \sin x} = \int \frac{2\cos x \sin x}{u} \frac{du}{-2\cos x \sin x}$$

$$= -\int \frac{1}{u}\,du = -\ln|u| + C$$

$$= -\ln|1-\cos^2 x| + C$$

(Double Substitution)

Example 11

(Old Exam Question) Find $\int x^3\sqrt{x^2+1}\,dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = x^2 + 1$$

$$du = 2xdx \rightarrow dx = \frac{du}{2x}$$

$$\int x^3\sqrt{x^2+1}\,dx = \int x^3\sqrt{u}\,\frac{du}{2x}$$
$$= \frac{1}{2}\int x^2\sqrt{u}\,du$$

Note that $x^2 = u - 1$.

$$
\begin{aligned}
&= \frac{1}{2}\int x^2\sqrt{u}\,du \\
&= \frac{1}{2}\int (u-1)u^{\frac{1}{2}}\,du \\
&= \frac{1}{2}\int u^{\frac{3}{2}} - u^{\frac{1}{2}} + C \\
&= \frac{1}{2}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + C \\
&= \frac{1}{5}(x^2+1)^{\frac{5}{2}} - \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C
\end{aligned}
$$

(Integral of $\tan x$)

Example 12

Find $\int \tan x \, dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. Note that $\tan x = \frac{\sin x}{\cos x}$. Hence we are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$\begin{aligned} u &= \cos x \\ du &= -\sin x dx \to dx = \frac{du}{-\sin x} \end{aligned}$$

$$\begin{aligned} \int \frac{\sin x}{\cos x} dx &= \int \frac{\sin x}{u} \frac{du}{-\sin x} = -\int \frac{1}{u} du \\ &= -\ln |u| + C \\ &= -\ln |\cos x| + C \end{aligned}$$

Example 13

Find $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = x^2$$

$$du = 2x\,dx \to dx = \frac{du}{2x}$$

$$\text{if } x = 0, \text{then } u = 0$$

$$\text{if } x = \sqrt{\pi}, \text{then } u = (\sqrt{\pi})^2 = \pi$$

$$\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = \int_0^{\pi} x\cos(u)\,\frac{du}{2x} = \frac{1}{2}\int_0^{\pi}\cos(u)\,du$$

$$= \left[\frac{1}{2}\sin(u)\right]_0^{\pi} = \left(\frac{1}{2}\sin(\pi)\right) - \left(\frac{1}{2}\sin(0)\right) = 0$$

Example 14

Find $\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. Let

$$u = \sin^{-1} x$$

$$du = \frac{1}{\sqrt{1-x^2}}\, dx \to dx = \sqrt{1-x^2}\, du$$

$$\text{if } x = 0, \text{then } u = \sin^{-1} 0 = 0$$

$$\text{if } x = \frac{1}{2}, \text{then } u = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$$

$$\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx = \int_0^{\frac{\pi}{6}} \frac{u}{\sqrt{1-x^2}} \sqrt{1-x^2}\, du = \int_0^{\frac{\pi}{6}} u\, du$$

$$= \left[\frac{1}{2}u^2\right]_0^{\frac{\pi}{6}} = \left(\frac{1}{2}\left(\frac{\pi}{6}\right)^2\right) - \left(\frac{1}{2}(0)^2\right) = \frac{\pi^2}{72}$$

Example 15

Find $\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. Let

$$u = e^x$$

$$du = e^x\,dx \rightarrow dx = \frac{du}{e^x}$$

$$\text{if } x = 0, \text{then } u = e^0 = 1$$

$$\text{if } x = \ln\sqrt{3}, \text{then } u = e^{\ln\sqrt{3}} = \sqrt{3}$$

$$\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx = \int_0^{\sqrt{3}} \frac{e^x}{1+(u)^2}\,\frac{du}{e^x} = \int_0^{\sqrt{3}} \frac{1}{1+(u)^2}\,du$$

$$= \left[\tan^{-1}(u)\right]_0^{\sqrt{3}} = \left(\tan^{-1}(\sqrt{3})\right) - \left(\tan^{-1}(0)\right) = \frac{\pi}{3}$$