Section 11.1
Definition of Derivative

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MATHS 104: Mathematics for Business II

Definition of the derivative

Recall: Derivative of a function $y = f(x)$ at any x is the slope of the tangent line at $(x, f(x))$.

$$\text{slope} = m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(z) - f(x)}{z - x}$$

If Q get closer and closer to P, the green line will get close and closer to the red line. The slope of the tangent line is given by

$$m = \lim_{z \to a} \frac{f(z) - f(x)}{z - x}$$

So the definition of the derivative is

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

"f prime of x".

Equivalent Definition

Recall the definition of the derivative is given

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

An equivalent definition (which is more *useful*) is given by setting $z = x + h$, hence as $z \to x$, we have $h \to 0$ and we get

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Example

(Old Exam Question) Use the definition of the derivative to find $f'(x)$ for $f(x) = 10 - 7x$.

Solution:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{10 - 7(x+h) - (10 - 7x)}{h} \\
&= \lim_{h\to 0} \frac{10 - 7x - 7h - 10 + 7x}{h} \\
&= \lim_{h\to 0} \frac{-7h}{h} \\
&= \lim_{h\to 0} -7 \\
&= -7
\end{aligned}
$$

Exercise

Using the definition of the limit, find the derivative of $f(x) = 3$. Can you generalize it to any constant function $f(x) = c$?

Exercise

(Homework) Using the definition of the limit, find the derivative of $f(x) = x$?

Exercise

(Homework) Using the definition of the limit, find the derivative of $f(x) = x^5$? (Hint: Use the first definition of the limit)

Example

(Old Exam Question) Use the definition of the derivative to find $f'(x)$ for $f(x) = x^2 - 8$.

Solution:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - 8 - (x^2 - 8)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 8 - x^2 + 8}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} \\
&= \lim_{h\to 0} 2x + h \\
&= 2x
\end{aligned}
$$

Example

Use the definition of the derivative to find $f'(x)$ for $f(x) = \sqrt{x+1}$.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+1} - (\sqrt{x+1})}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{(\sqrt{x+h+1} + \sqrt{x+1})}{(\sqrt{x+h+1} + \sqrt{x+1})} \\
&= \lim_{h\to 0} \frac{x+h+1-x-1}{h(\sqrt{x+h+1} + \sqrt{x+1})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})} \\
&= \lim_{h\to 0} \frac{1}{(\sqrt{x+h+1} + \sqrt{x+1})} = \frac{1}{2\sqrt{x+1}}
\end{aligned}$$

Exercise

(Homework) Using the definition of the limit, find the derivative of $f(x) = \sqrt{x}$?

Example

Use the definition of the derivative to find $f'(x)$ for $f(x) = \frac{6}{x}$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{6}{x+h} - (\frac{6}{x})}{h} \\ &= \lim_{h\to 0} \frac{\frac{6x-6(x+h)}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{-6h}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{-6}{x(x+h)} \\ &= \frac{-6}{x^2} \end{aligned}$$

Example

Find the equation of the tangent line to the curve $f(x) = \frac{6}{x}$ at $x = 3$.

Solution: To find the equation of the tangent line, we need to find the slope of the tangent line. From the previous example, we found that

$$f'(x) = \frac{-6}{x^2}$$

The slope is the derivative at $x = 3$, is hence

$$m = f'(3) = \frac{-6}{3^2} = -\frac{2}{3}$$

The equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{-2}{3}(x - 3)$$

$$2x + 3y = 12$$

Exercise

(Homework) Using the definition of the limit, find the derivative of $f(x) = \frac{1}{x}$?

Other Notation

- $\frac{dy}{dx}$ "dee y, dee x" or "dee y by dee x".
- $\frac{d}{dx}(f(x))$ "dee $f(x)$, dee x" or "dee $f(x)$ by dee x".
- y' "y prime".
- $\frac{dy}{dx}_{x=a}$ or $y'(a)$ means $f'(a)$.