Section 11.3
The Derivative as a Rate of Change

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MATHS 104: Mathematics for Business II

Rate of change

Recall:

- If a line has a slope $m = 3$, it means that for every one step to the right, we move along the line 3 steps up. In this case, as x increases, y increases.
- If a line has a slope $m = -2$, it means that for every 1 step to the right, we move along the line 2 steps down. In this case, as x increases, y decreases.

For general function $y = f(x)$, for every step to the right, how many steps to go up/down? How do we measure that change in y?
If x changes by 1, an estimate of the change in y is $\frac{dy}{dx}$.

Definition

The derivative of $y = f(x)$ can be interpreted as *rate of change* of y in term of x.

Definition

Let $y = f(x)$ be a function, then

- The **rate of change** of $f(x)$ is

$$f'(x)$$

- The **relative rate of change** of $f(x)$ is

$$\frac{f'(x)}{f(x)}$$

- The **percentage rate of change** of $f(x)$ is

$$\frac{f'(x)}{f(x)} \cdot 100\%$$

Example

(Old Final Exam Question) It is projected that x months from now, the population of a certain town will be $P(x) = 2x + 4x^{\frac{3}{2}} + 5000$. At what percentage rate of change will the population be changing with respect to time 9 months from now?

Solution:

$$\text{Percentage rate} = \frac{P'(x)}{P(x)} \cdot 100\%$$
$$= \frac{2 + 6x^{\frac{1}{2}}}{2x + 4x^{\frac{3}{2}} + 5000} \cdot 100\%$$

Now we substitute $x = 9$ to get

$$\text{Percentage rate} = 0.1288\%$$

Exercise

(Old Exam Question) Consider the cost function $c(q) = 1.3q^2 + 0.2q - 8$. Determine the percentage rate of change of c with respect to q when $q = 10$.

Marginal Cost

Recall: The **total–cost** of any manufacturer is calculated based on the quantity that is being produced. usually, we write this as

$$c = f(q)$$

Definition

The rate of change of c with respect to q is called **marginal cost**,

$$\text{marginal cost} = \frac{dc}{dq}$$

Definition

The **average cost** per unit for a total cost function is given by

$$\bar{c} = \frac{c}{q}$$

Note: $c = q\bar{c}$.

Example

(Old Exam Question) Find the marginal cost function if the average cost function is

$$\overline{c} = 2q + \frac{10000}{q^2}$$

Solution: Recall that

$$\text{marginal cost} = \frac{dc}{dq}$$

We need first to find the cost function which is given by

$$\begin{aligned} c(q) = q\overline{c} &= q\left(2q + \frac{10000}{q^2}\right) \\ &= 2q^2 + \frac{10000}{q} \\ &= 2q^2 + 10000q^{-1} \end{aligned}$$

hence,

$$\text{marginal cost} = \frac{dc}{dq} = 4q - 10000q^{-2}$$

Exercise

Find the marginal cost function if the average cost function is

$$\overline{c} = 0.002q^2 - 0.5q + 60 + \frac{7000}{q}$$

Find the marginal cost for $q = 15$ and $q = 25$

Example

Find the marginal revenue function if the revenue function is

$$r = 2q(30 - 0.1q)$$

Find the marginal revenue at $q = 10$, and $q = 20$.

Solution: Recall that

$$\text{marginal revenue} = \frac{dr}{dq}$$

We need first to rewrite the revenue function which is given by

$$\begin{aligned} r(q) &= 2q\left(30 - 0.1q\right) \\ &= 60q - 0.2q^2 \end{aligned}$$

hence,

$$\text{marginal revenue} = \frac{dr}{dq} = 60 - 0.4q$$

Continue...

To find the marginal revenue at $q = 10$ and $q = 20$, we substitute in the derivative to get

$$\text{marginal revenue} = \frac{dr}{dq}_{q=10} = 60 - 0.4(10) =$$
$$\text{marginal revenue} = \frac{dr}{dq}_{q=20} = 60 - 0.4(20) =$$

Exercise

Find the marginal revenue function if the revenue function is

$$r(q) = 240q + 40q^2 - 2q^3$$

Find the marginal cost for $q = 15$ and $q = 25$