Section 11.5
The Chain Rule

Dr. Abdulla Eid

College of Science

MATHS 104: Mathematics for Business II

Motivation

Goal: We want to derive rules to find the derivative of composite of two functions $f(g(x))$

Example

We want to find (in a general way) the derivative of the functions (Note the inner and the outer functions)

- $f(x) = (3x^2 + 5x + 1)^3 = (\underbrace{3x^2 + 5x + 1}_{inner})^{\overbrace{3}^{outer}}$
- $f(x) = (2x^3 - 8x)^{\frac{-4}{3}} = (\underbrace{2x^3 - 8x}_{inner})^{\overbrace{\frac{-4}{3}}^{outer}}$
- $f(x) = \frac{4}{x^2+5} = = 4(x^2 + 5)^{-1} = (\underbrace{x^2 + 5}_{inner})^{\overbrace{-1}^{outer}}$

The Chain Rule

Theorem

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

$$(f(g(x)))' = \textit{derivative of outer}(\textit{inner}) \cdot (\textit{derivative of inner})$$

Example

Find the derivative of each of the following:

1. $f(x) = (3x^2 + 5x + 1)^3$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.

(1) $f(x) = (3x^2 + 5x + 1)^3 = (3x^2 + 5x + 1)^3$.

$$\begin{aligned} f'(x) &= \text{derivative of outer}\,(\text{inner}) \cdot (\text{derivative of inner}) \\ &= 3(3x^2 + 5x + 1)^2 \cdot (6x + 5) \end{aligned}$$

Example

Find the derivative of each of the following:

(2) $f(x) = (2x^3 - 8x)^{\frac{-4}{3}}$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.

(2) $f(x) = (2x^3 - 8x)^{\frac{-4}{3}} = (2x^3 - 8x)^{\frac{-4}{3}}$.

$$
\begin{aligned}
f'(x) &= \text{derivative of outer}\,(\text{inner}) \cdot (\text{derivative of inner}) \\
&= \frac{-4}{3}(2x^3 - 8x)^{\frac{-7}{3}} \cdot (6x^2 - 8)
\end{aligned}
$$

Example

Find the derivative of each of the following:

(3) $f(x) = \frac{4}{x^2+5}$

Solution: We write the inner function in blue and the outer function in red and we apply the chain rule.

(3) $f(x) = \frac{4}{x^2+5} = = 4(x^2+5)^{-1} = 4(x^2+5)^{-1}$.

$$f'(x) = \text{derivative of outer (inner)} \cdot (\text{derivative of inner})$$
$$= -4(x^2+5)^{-2} \cdot (2x)$$

General Power Rule

Theorem

The general power rule

$$\frac{d}{dx}(u)^n = nu^{n-1} \cdot u'$$

Example

(Old Exam Question) Find the derivative of each of the following:

1. $F(x) = \sqrt[3]{4 - 5x^6}$.

Solution: Re-write the function as $f(x) = (4 - 5x^6)^{\frac{1}{3}}$ and apply the general power rule.

$$f'(x) = \frac{1}{3}(4 - 5x^6)^{-\frac{2}{3}} \cdot (-30x^5)$$

Example

Suppose the $p = 100 - \sqrt{q^2 + 20}$ is a demand function for a manufacturer's product. Find

(a) The rate of change of p with respect to q.

(b) The relative rate of change of p with respect to q.

(c) Find the marginal revenue function.

Solution: We first re–write the function as

$p = 100 - \sqrt{q^2 + 20} = 100 - (q^2 + 20)^{\frac{1}{2}}$.

(a) The rate of change is

$$p' = -\frac{1}{2}(q^2 + 20)^{-\frac{1}{2}} \cdot (2q)$$

$$f'(x) = \frac{-q}{\sqrt{q^2 + 20}}$$

Continue

Example

Suppose the $p = 100 - \sqrt{q^2 + 20}$ is a demand function for a manufacturer's product. Find
(b) The relative rate of change of p with respect to q.
(c) Find the marginal revenue function.

Solution: We first re–write the function as
$p = 100 - \sqrt{q^2 + 20} = 100 - (q^2 + 20)^{\frac{1}{2}}$.
(b) The relative rate of change is

$$\frac{p'}{p} = \frac{\frac{-q}{\sqrt{q^2+20}}}{100 - \sqrt{q^2 + 20}}$$

Continue

Example

Suppose the $p = 100 - \sqrt{q^2 + 20}$ is a demand function for a manufacturer's product. Find
(c) Find the marginal revenue function.

Solution:
(c) The revenue function is $r(q) = qp$, hence the marginal derivative is

$$r'(x) = p + qp'$$
$$f'(x) = 100 - \sqrt{q^2 + 20} + q\frac{-q}{\sqrt{q^2 + 20}}$$

Extra Exercises

Exercise

1. $y = \sqrt{2x} + \frac{1}{\sqrt{2x}}$.
2. $y = \left(\frac{x+1}{x+2}\right)^2$.
3. $y = (3x+5)^5(2x^2 - 3x + 5)^3$.
4. $y = \frac{4}{\sqrt{9x^2+1}}$.