Section 12.2
The Derivative of Logarithmic and Exponential Functions togetrher with the Chain Rule

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MATHS 104: Mathematics for Business II

Recall

The Chain Rule

Theorem

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

$$(f(g(x)))' = \textit{derivative of outer}\,(\textit{inner}) \cdot (\textit{derivative of inner})$$

Theorem

- $(\ln x)' = \frac{1}{x}$.
- $(e^x)' = e^x$

Example

(Old Exam Question) Find the derivative of

1. $f(x) = \log(1+x^2)$

Solution: First we re–write the function in terms of ln to get

$$f(x) = \frac{\ln(1+x^2)}{\ln 10} = \frac{1}{\ln 10}\ln(1+x^2)$$

Next we write the inner function in blue and the outer function in red and we apply the chain rule.

$$f(x) = \frac{1}{\ln 10}\ln(1+x^2) = \frac{1}{\ln 10}\ln(1+x^2)$$

$$\begin{aligned} f'(x) &= \text{derivative of outer (inner)} \cdot (\text{derivative of inner}) \\ &= \frac{1}{\ln 10}\frac{1}{1+x^2} \cdot (2x) \\ &= \frac{2x}{\ln(10)(1+x^2)} \end{aligned}$$

Exercise

Find the derivative of:

$$f(x) = \log(x^2 - 5)$$

Example

(Old Exam Question) Find the derivative of:

$$f(x) = (3-x)^4 e^{2x}$$

Solution: Here we use the product rule first to get

$$\begin{aligned} f'(x) &= (\text{derivative of first})\,(\text{second}) + (\text{first})(\text{derivative of second}) \\ &= ((3-x)^4)'(e^{2x}) + ((3-x)^4)(e^{2x})' \end{aligned}$$

We write the inner function in blue and the outer function in red and we apply the chain rule.

$$\text{derivative of outer}\,(\text{inner}) \cdot (\text{derivative of inner})$$

$$f'(x) = ((3-x)^4)'(e^{2x}) + ((3-x)^4)(e^{2x})'$$

$$f'(x) = (4(3-x)^3 \cdot -1)(e^{2x}) + ((3-x)^4)(e^{2x} \cdot 2)$$

Example

(Old Exam Question) Find the derivative of:

$$f(x) = (1+x^2)^5 \ln(1+x^2)$$

Solution: Here we use the product rule first to get

$$\begin{aligned} f'(x) &= (\text{derivative of first})\,(\text{second}) + (\text{first})(\text{derivative of second}) \\ &= ((1+x^2)^5)'(\ln(1+x^2)) + ((1+x^2)^5)(\ln(1+x^2))' \end{aligned}$$

We write the inner function in blue and the outer function in red and we apply the chain rule.

$$\text{derivative of outer}\,(\text{inner}) \cdot (\text{derivative of inner})$$

$$f'(x) = ((1+x^2)^5)'(\ln(1+x^2)) + ((1+x^2)^5)(\ln(1+x^2))'$$

$$f'(x) = (5(1+x^2)^4 \cdot 2x)(\ln(1+x^2)) + ((1+x^2)^5)(\frac{1}{1+x^2} \cdot 2x)$$

$$f'(x) = 10x(1+x^2)^4 \ln(1+x^2) + 2x(1+x^2)^4$$

Example

(Old Final Exam Question) Find the derivative of

1. $f(x) = (x - x\ln x)^{104}$

Solution: Next we write the inner function in blue and the outer function in red and we apply the chain rule. Note that we use the product rule to differentiate the inner function

$$f(x) = (x - x\ln x)^{104}$$

$$\begin{aligned} f'(x) &= \text{derivative of outer (inner)} \cdot (\text{derivative of inner}) \\ &= 104(x - x\ln x)^{103} \cdot (x - x\ln x)' \\ &= 104(x - x\ln x)^{103} \cdot (1 - \ln x - x\frac{1}{x}) \\ &= 104(x - x\ln x)^{103} \cdot (-\ln x) \end{aligned}$$

Derivative using the properties of Logarithms

Example

(Old Final Exam Question) Find the derivative of

1. $f(x) = \ln x^{2016}$

Solution: First we re–write the function in terms using the properties of the ln to get a simplified function:

$$f(x) = 2016 \ln x$$

Hence

$$f'(x) = 2016 \frac{1}{x}$$

Exercise

Using the chain rule, find the derivative of the function of the previous example *without using the properties of the* ln, i.e., find $f'(x)$ for

$$f(x) = \ln(x^{2016})$$

Derivative using the properties of Logarithms

Example

(Old Final Exam Question) Find the derivative of

1. $f(x) = \ln \sqrt[3]{\frac{x^3-1}{x^3+1}}$

Solution: First we re–write the function in terms using the properties of the ln to get a simplified function:

$$f(x) = \left(\frac{x^3-1}{x^3+1}\right)^{\frac{1}{3}}$$
$$= \frac{1}{3}\left(\ln(x^3-1) - \ln(x^3+1)\right)$$

Continue...

We write the inner function in blue and the outer function in red and we apply the chain rule.

derivative of outer (inner) · (derivative of inner)

$$f(x) = \frac{1}{3}\left(\ln(x^3 - 1) - \ln(x^3 + 1)\right)$$
$$f'(x) = \frac{1}{3}\left(\frac{1}{x^3 - 1} \cdot (3x^2) - \frac{1}{x^3 + 1} \cdot (3x^2)\right)$$

Exercise

Using the chain rule, find the derivative of the function of the previous example *without using the properties of the* ln, i.e., find $f'(x)$ for

$$f(x) = \ln\left(\sqrt[3]{\frac{x^3-1}{x^3+1}}\right)$$

Example

(Old Final Exam Question) If $c(q) = 500 + 0.001e^{2q+5}$ is the cost function. Find the marginal cost (rounded to two decimal places) when $q = 5$.

Solution: The marginal cost is $c'(q)$. To find it we need to write the inner function in blue and the outer function in red and we apply the chain rule.

$$f(x) = 500 + 0.001e^{2q+5}$$

$$c'(x) = \text{derivative of outer}\,(\text{inner}) \cdot (\text{derivative of inner})$$

$$c'(q) = 0.001e^{2q+5} \cdot (2)$$

$$c'(5) = 0.001e^{2(5)+5} \cdot (2)$$

$$c'(5) = 6538.03$$