Section 13.3
Concavity and Curve Sketching

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MATHS 104: Mathematics for Business II

Concavity

Increasing Function has three cases

Question: How to distinguish between these three types of behavior?
Answer: Recall: If $g(x)$ is increasing, then $g'(x) > 0$.

1. If $f''(x) > 0$, then the curve is concave upward (CU).
2. If $f''(x) < 0$, then the curve is concave downward (CD).
3. If $f''(x) = 0$ (for all x), then $f(x)$ has no curvature (line).

Inflection Points

Definition

A number c is called an inflection point of $f(x)$ if at these point, the function changes from concave upward to downward and vice verse. The candidates are the points c, where

$$f''(c) = 0 \text{ or } f''(c) \text{ does not exist}$$

Example

Discuss the following curve with respect to concavity and inflection points.

$$f(x) = x^3$$

Solution:

We find the derivatives first which are

$$f'(x) = 3x^2$$
$$f''(x) = 6x$$

To find the inflection points, we find where the second derivative equal to zero or does not exist.

$$f''(x) = 0$$
$$\text{numerator} = 0$$
$$6x = 0$$
$$x = 0$$

$f'(x)$ does not exist

$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is CD in $(-\infty, 0)$.
2. f is CU in $(0, \infty)$.
3. f has inflection point at $x = 0$ with value $f(0) = 0$.

Example

Discuss the following curve with respect to concavity and inflection points.

$$f(x) = 2 + \ln x$$

Solution:
We find the derivatives first which are

$$f'(x) = \frac{1}{x}$$
$$f''(x) = \frac{-1}{x^2}$$

To find the inflection points, we find where the second derivative equal to zero or does not exist.

$$f''(x) = 0$$
$$\text{numerator} = 0$$
$$-1 = 0$$

Always False

No Solution

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$x^2 = 0$$
$$x = 0$$

Number Line

1. f is CD in $(0, \infty)$.
2. f has no inflection point.

Example

(Old Exam Question) Discuss the following curve with respect to concavity and inflection points.

$$f(x) = x^4 - 3x^3 + 3x^2 - 5$$

Solution:

We find the derivatives first which are

$$f'(x) = 4x^3 - 9x^2 + 6x$$

$$f''(x) = 12x^2 - 18x + 6$$

To find the inflection points, we find where the second derivative equal to zero or does not exist.

$$f''(x) = 0$$

$$\text{numerator} = 0$$

$$12x^2 - 18x + 6 = 0$$

$$f'(x) \text{ does not exist}$$

$$\text{denominator} = 0$$

$$1 = 0$$

Always False

Number Line

1. f is CD in $(\frac{1}{2}, 1)$.
2. f is CU in $(-\infty, \frac{1}{2}) \cup (1, \infty)$.
3. f has inflection point at $x = \frac{1}{2}$ with value $f(\frac{1}{2}) =$ and at $x = 1$ with value $f(1) =$.

Exercise

(All in All) Find the intervals where the function is increasing/decreasing, concave upward, concave downward, find all local max/min, find inflection points and sketch the graph of the function.

$$f(x) = x^5 - 4x^4$$

Solution: We find the derivative first which is

$$f'(x) = 5x^4 - 20x^3$$
$$f''(x) = 20x^3 - 60x^2$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$
$$\text{numerator} = 0$$
$$5x^4 - 20x^3 = 0$$
$$x = 0 \text{ or } x = 4$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is increasing in $(-\infty, 0) \cup (4, \infty)$.
2. f is decreasing in $(0, 4)$.
3. f has a local maximum at $x = 0$ with value $f(0) = 0$.
4. f has a local minimum at $x = 4$ with value $f(4) =$.

Recall that the derivatives are

$$f'(x) = 5x^4 - 20x^3$$
$$f''(x) = 20x^3 - 60x^2$$

To find the inflection points, we find where the second derivative equal to zero or does not exist.

$f''(x) = 0$

numerator $= 0$

$20x^3 - 60x^2 = 0$

$x = 1$ or $x = 3$

$f'(x)$ does not exist

denominator $= 0$

$1 = 0$

Always False

No Solution

Number Line

1. f is CD in $(-\infty, 3)$.
2. f is CU in $(3, \infty)$.
3. f has inflection point at $x = 3$ with value $f(3) =$.

Exercise

(All in All) Find the intervals where the function is increasing/decreasing, concave upward, concave downward, find all local max/min, find inflection points and sketch the graph of the function.

$$f(x) = x^3 - 6x^2 + 9x + 1$$

Solution: We find the derivative first which is

$$f'(x) = 3x^2 - 12x + 9$$
$$f''(x) = 6x - 12$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$
$$\text{numerator} = 0$$
$$3x^2 - 12x + 9 = 0$$
$$x = 1 \text{ or } x = 3$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is increasing in $(-\infty, 1) \cup (3, \infty)$.
2. f is decreasing in $(1, 3)$.
3. f has a local maximum at $x = 1$ with value $f(1) = 5$.
4. f has a local minimum at $x = 3$ with value $f(3) = 1$.

Recall that the derivatives are

$$f'(x) = 3x^2 - 12x + 9$$
$$f''(x) = 6x - 12$$

To find the inflection points, we find where the second derivative equal to zero or does not exist.

$$f''(x) = 0$$
$$\text{numerator} = 0$$
$$6x - 12 = 0$$
$$x = 2$$

$f'(x)$ does not exist

$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is CD in $(-\infty, 2)$.
2. f is CU in $(2, \infty)$.
3. f has inflection point at $x = 2$ with value $f(2) = 3$.