Section 14.4
More Integration Formula (The Substitution Method)
2 Lectures

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The Substitution Method

Idea: To replace a relatively complicated integral by a simpler one (one from the list). This is done by adding an extra variable we call it u.

Theorem 1

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$$

Two properties we are looking for in u:

1. u should be an inner function.
2. Almost the derivative of u appears in the integral.

Example 2

Find $\int xe^{x^2}\,dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = x^2$$

$$du = 2xdx \to dx = \frac{du}{2x}$$

$$\begin{aligned}\int xe^{x^2}\,dx &= \int xe^{u}\,\frac{du}{2x}\\ &= \frac{1}{2}\int e^{u}\,du\\ &= \frac{1}{2}e^{u} + C\\ &= \frac{1}{2}e^{x^2} + C\end{aligned}$$

Exercise 3

Find $\int \sqrt{3x+5}\, dx$.

Example 4

Find $\int \frac{x^2}{\sqrt{1-x^3}}\, dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = 1 - x^3$$

$$du = -3x^2 dx \rightarrow dx = \frac{du}{-3x^2}$$

$$\int \frac{x^2}{\sqrt{1-x^3}}\, dx = \int \frac{x^2}{\sqrt{u}} \frac{du}{-3x^2} = \frac{1}{-3} \int \frac{1}{\sqrt{u}}\, du$$
$$= \frac{1}{-3} \int (u)^{-\frac{1}{2}}\, du = \frac{2}{-3} u^{\frac{1}{2}} + C$$
$$= \frac{2}{-3} (1 - x^3)^{\frac{1}{2}} + C$$

Exercise 5

Find $\int e^x \sqrt{1+e^x}\, dx$.

Example 6

Find $\int \frac{(\ln x)^2}{x}\, dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = \ln x$$

$$du = \frac{1}{x} dx \rightarrow dx = x du$$

$$\int \frac{(\ln x)^2}{x}\, dx = \int \frac{(u)^2}{x} x du = \int (u)^2\, du$$

$$= \frac{1}{3} u^3 + C$$

$$= \frac{1}{3} (\ln x)^3 + C$$

Exercise 7

Find $\int \frac{1}{x\sqrt{\ln x}}\,dx$.

Exercise 8

(Old Exam Question) Find $\int x\sqrt{3-4x^2}\,dx$.

Example 9

Find $\int \frac{1}{ax+b}\, dx (a \neq 0)$

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = ax + b$$

$$du = adx \to dx = \frac{du}{a}$$

$$\int \frac{1}{ax+b} dx = \int \frac{1}{u} \frac{du}{a} = \frac{1}{a} \int \frac{1}{u}\, du$$

$$= \frac{1}{a} \ln u + C$$

$$= \frac{1}{a} \ln(ax + b) + C$$

Example 10

(Old Exam Question) Find $\int e^{9x}\,dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = 9x$$

$$du = 9dx \to dx = \frac{du}{9}$$

$$\begin{aligned}\int e^{9x}\,dx &= \int e^{u}\,\frac{du}{9} \\ &= \frac{1}{9}\int e^{u}\,du \\ &= \frac{1}{9}e^{u} + C \\ &= \frac{1}{9}e^{9x} + C\end{aligned}$$

Basic Rules:

4 $\int e^x\,dx = e^x + C \rightarrow \int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$

Example 11

1. $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C.$
2. $\int e^{-x}\,dx = \frac{1}{-1}e^{-x} + C.$
3. $\int e^{-2x}\,dx = \frac{1}{-2}e^{-2x} + C.$

Example 12

(Old Exam Question) Find $\int (e^x + 5)^2 \, dx$.

Solution: Can we use the substitution method? No, the derivative is not almost there!. We expand it

$$\int (e^x + 5)^2 \, dx = \int (e^{2x} + 10e^x + 25) \, dx$$
$$= \frac{1}{2} e^{2x} + 10e^x + 25x + C$$

Exercise 13

(Old Exam Question) Find $\int (e^x - e^{-x})^2 \, dx$.

Example 14

(Old Exam Question) Find $\int \frac{1+e^x}{e^x}\, dx$.

Solution:

$$\begin{aligned}\int \frac{1+e^x}{e^x}\, dx &= \int\left(\frac{1}{e^x} + \frac{e^x}{e^x}\right)\, dx \\ &= \int (e^{-x}+1)\, dx \\ &= -e^{-x} + x + C\end{aligned}$$

Example 15

(Old Final Exam Question) Find $\int \frac{5x^2}{(7-2x^3)^4}\, dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = 7 - 2x^3$$

$$du = -6x^2 dx \rightarrow dx = \frac{du}{-6x^2}$$

$$\int \frac{5x^2}{(7-2x^3)^4}\, dx = \int \frac{5x^2}{u^4} \frac{du}{-6x^2} = \frac{5}{-6} \int \frac{1}{u^4}\, du$$

$$= \frac{5}{-6} \int (u)^{-4}\, du = \frac{5}{18} u^{-3} + C$$

$$= \frac{5}{18} (7-2x^3)^{-3} + C$$

Exercise 16

(Old Exam Question) Find $\int 7x^3 e^{1-x^4}\, dx$.

Example 17

Find $\int 9000x^8(999 - x^9)^{999}\, dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = 999 - x^9$$

$$du = -9x^8 dx \to dx = \frac{1}{-9x^8} du$$

$$\begin{aligned}\int 9000x^8(999 - x^9)^{999}\, dx &= \int 9000x^8(u)^{999} \frac{1}{-9x^8} du \\ &= -1000 \int (u)^{999}\, du \\ &= -u^{1000} + C \\ &= -(999 - x^9)^{1000} + C\end{aligned}$$

Exercise 18

(Old Exam Question) Find $\int \frac{x^5}{e^{x^6}}\, dx$.

Example 19

Find $\int (3x^2+2)\sqrt{2x^3+4x+1}\,dx$

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = 2x^3 + 4x + 1$$

$$du = (6x^2+4)dx \rightarrow dx = \frac{du}{(6x^2+4)}$$

$$\begin{aligned}\int (3x^2+2)\sqrt{2x^3+4x+1}\,dx &= \int (3x^2+2)\sqrt{u}\,\frac{du}{6x^2+4} \\ &= \int (3x^2+2)\sqrt{u}\,\frac{du}{2(3x^2+2)} \\ &= \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\int u^{\frac{1}{2}}\,du \\ &= \frac{1}{3}(u)^{\frac{3}{2}} + C = \frac{1}{3}(2x^3+4x+1)^{\frac{3}{2}} + C\end{aligned}$$

Exercise 20

(Old Exam Question) Find $\int 2x(5 - e^{x^2/8})\, dx$.

Hint: $\int 2x(5 - e^{x^2/8})\, dx = \int (10x - 2xe^{x^2/8})\, dx$. Now use the substitution method.

Example 21

(Old Exam Question) The marginal revenue is $r'(q) = 60 - 0.04q - \frac{200}{(q+1)^2}$. Find the revenue function.

Solution: We integrate to find the revenue r first. Note that we have used the substitution method.

$$r = \int \left(60 - 0.04q - \frac{200}{(q+1)^2}\right) dq$$

$$r = 60q - 0.02q^2 + 200(q+1)^{-1} + C$$

$$0 = r(0) \qquad \textbf{because no revenue if no quanitity is sold}$$

$$0 = 240(0) - 2(0)^2 + 200 + C$$

$$-200 = C$$

$$r = 60q - 0.02q^2 + 200(q+1)^{-1} - 200$$

Example 22

If y satisfies $y' = \frac{x}{x^2+6}$ and $y(1) = 0$. Find y.

Solution: Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = x^2 + 6$$

$$du = 2x dx \to dx = \frac{du}{2x}$$

$$\int \frac{x}{x^2+6}\, dx = \int \frac{x}{u} \frac{du}{2x} = \frac{1}{2} \int \frac{1}{u}\, du$$

$$= \frac{1}{2} \ln u + C = \frac{1}{2} \ln(x^2+6) + C$$

$$0 = \frac{1}{2} \ln(1+6) + C \to C = -\frac{1}{2} \ln 7$$

$$y = \frac{1}{2} \ln(x^2+6) - \frac{1}{2} \ln 7$$