Section 14.7
Definite Integrals
2 Lectures

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MATHS 104: Mathematics for Business II

Definite Integral

Recall: The integral is used to find area under the curve over an interval $[a, b]$

Idea: To cover the area by as many rectangles as possible and then we will get better and better estimate if we increase the number of rectangles.

Question: When will we get an exact estimate for the area?

Answer: When the number of rectangle $\to \infty$. In that case, we write the area by

$$\text{Area} = \int_a^b f(x)\, dx$$

This integral is called definite integral. The number a and b are called the *lower limit and upper limit of integration* respectively.

The Fundamental Theorem of Calculus

Question: How to evaluate the definite integral?

Theorem 1

If f is continuous on the interval $[a, b]$ and F is the anti-derivative of f, then

$$\int_a^b f(x)\,dx = \Bigg[\underbrace{F(x)}_{antiderivative}\Bigg]_a^b = F(b) - F(a)$$

1. Definite integral $\int_a^b f(x)\,dx$ gives a **number** represents the area.
2. Indefinite integral $\int f(x)\,dx$ gives a **function**.

Example 2

Find $\int_{-1}^{2} (x^3 - 6x)\, dx$.

Solution: [1]

$$\int_{-1}^{2} (x^3 - 6x)\, dx = \int_{-1}^{2} (x^3 - 6x)\, dx = \left[\frac{1}{4}x^4 - 3x^2\right]_{-1}^{2}$$

$$\left(\frac{1}{4}(2)^4 - 3(2)^2\right) - \left(\frac{1}{4}(-1)^4 - 3(-1)^2\right) = \frac{-21}{4}$$

[1] Direct evaluation

Example 3

Find $\int_1^9 6\sqrt{x}\, dx$.

Solution: [2]

$$\int_1^9 6\sqrt{x}\, dx = \int_1^9 6x^{\frac{1}{2}}\, dx = \left[6\frac{2}{3}x^{\frac{3}{2}}\right]_1^9$$
$$\left(4(9)^{\frac{3}{2}}\right) - \left(4(1)^{\frac{3}{2}}\right) = 104$$

[2] Direct evaluation

Exercise 4

Find $\int_{-1}^{1}(x+1)^2\,dx$.

Example 5

Find $\int_1^2 \frac{x^5+3x^3}{x^4}\,dx$.

Solution: [3]

$$\int_1^2 \frac{x^5+3x^3}{x^4}\,dx = \int_1^2 x + \frac{3}{x}\,dx = \left[\frac{1}{2}x^2 + 3\ln|x|\right]_1^2$$

$$\left(\frac{1}{2}(2)^2 + 3\ln 2\right) - \left(\frac{1}{2}(1)^2 + 3\ln 1\right) = \frac{3}{2} + 3\ln 2$$

[3] Direct evaluation

Example 6

(Substitution and definite integrals) Find $\int_0^1 x^2 e^{-x^3}\,dx$

Solution: [4] Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = -x^3$$

$$du = -3x^2\,dx \to dx = \frac{du}{-3x^2}$$

$$\text{if } x = 0, \text{then } u = 0$$

$$\text{if } x = 1, \text{then } u = -1$$

$$\int_0^1 x^2 e^{-x^3}\,dx = \int_0^{-1} x^2 e^u \frac{du}{-3x^2} = \frac{1}{-3}\int_0^{-1} e^u\,du$$

$$= \left[\frac{-1}{3}e^u\right]_0^{-1} = = \left(\frac{-1}{3}e^{-1}\right) - \left(\frac{-1}{3}e^0\right) = -\frac{1}{3}e^{-1} + \frac{1}{3}$$

[4] Substitution and then evaluation

Example 7

(Substitution and definite integrals) Find $\int_0^4 \frac{-5x}{\sqrt{x^2+9}}\,dx$

Solution: [5]Since this is not a basic integral, we are looking for a good substitution. We are looking for an inner function with almost the derivative is somewhere in the integral. Let

$$u = x^2 + 9$$

$$du = 2x\,dx \rightarrow dx = \frac{du}{2x}$$

$$\text{if } x = 0, \text{then } u = 9$$

$$\text{if } x = 4, \text{then } u = 25$$

$$\int_0^4 \frac{-5x}{\sqrt{x^2+9}}\,dx = \int_9^{25} \frac{-5x}{\sqrt{u}}\,\frac{du}{2x} = \frac{-5}{2}\int_9^{25} \frac{1}{\sqrt{u}}\,du$$

$$= \frac{-5}{2}\int_9^{25} (u)^{-\frac{1}{2}}\,du = \left[-5u^{\frac{1}{2}}\right]_9^{25}$$

Example 8

(Old Final Exam Question) If $\int_a^3 (3x^2 + 2x)\,dx = 36$, then find the value of a.

Solution: [6]

$$\begin{aligned}
36 &= \int_a^2 (3x^2 + 2x)\,dx = \int_a^3 (3x^2 + 2x)\,dx = \left[x^3 + x^2\right]_a^3 \\
36 &= (36) - \left(a^3 + a^2\right) \\
36 &= -a^3 - a^2 + 36 \\
0 &= -a^3 - a^2 \\
0 &= -a^2(a + 1) \\
a &= 0 \text{ or } a = -1
\end{aligned}$$

[6] Finding limit of integration

Example 9

(Old Final Exam Question) If $\int_a^2 (x+1)^2\,dx = 9$, then find the value of a.

Solution: [7]

$$9 = \int_a^2 (x+1)^2\,dx = \int_a^2 (x^2+2x+1)\,dx = \left[\frac{1}{3}x^3 + x^2 + x\right]_a^2$$

$$9 = \left(\frac{26}{3}\right) - \left(\frac{1}{3}a^3 + a^2 + a\right)$$

$$9 = -\frac{1}{3}a^3 - a^2 - a + \frac{26}{3}$$

$$0 = -\frac{1}{3}a^3 - a^2 - a + \frac{-2}{3}$$

$$a = -2$$

[7] Finding limit of integration

Properties of Integration

Recall: Definite integrals compute the area under the curve, i.e.,

$$\text{Area} = \int_a^b f(x)\,dx$$

1. $\int_a^b [c \cdot f(x)]\,dx = c \cdot \int_a^b f(x)\,dx.$
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$
3. $\int_a^a f(x)\,dx = 0.$
4. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$
5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b g(x)\,dx.$
6. If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$

Example 10

(Old Final Exam Question) If $\int_0^2 f(x)\,dx = 3$, $\int_0^2 g(x)\,dx = 2$, then find $\int_0^2 [4f(x) + g(x)]\,dx$.

Solution: [8]

$$\begin{aligned}\int_0^2 [4f(x) + g(x)] &= 4\int_0^2 [f(x)\,dx] + \int_0^2 [g(x)]\,dx \\ &= 4(3) + 2 \\ &= 14\end{aligned}$$

[8] Properties of integral

Exercise 11

If $\int_1^5 f(x)\,dx = 3$, $\int_1^3 f(x)\,dx = 1$, and $\int_1^3 h(x)\,dx = 5$ then find [a]

1. $\int_1^5 -2f(x)\,dx$.
2. $\int_1^3 [f(x) + h(x)]\,dx$.
3. $\int_1^3 [2f(x) - 5h(x)]\,dx$.
4. $\int_5^1 f(x)\,dx$.
5. $\int_3^5 f(x)\,dx$.
6. $\int_3^1 [h(x) - f(x)]\,dx$.
7. $\int_3^3 [h(x) - f(x)]\,dx$.

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[a] Properties of integral

Example 12

(Old Final Exam Question) Given

$$f(x) = \begin{cases} 4x+2, & x < 2 \\ 3x^2 - 2, & 2 \le x < 6 \\ 106, & x \ge 6 \end{cases}$$

Evaluate $\int_0^4 f(x)\,dx$

Solution: [9]

$$\begin{aligned} \int_0^4 f(x)\,dx &= \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx \\ &= \int_0^2 4x+2\,dx + \int_2^4 3x^2 - 2\,dx \\ &= \left[2x^2 + 2x\right]_0^2 + \left[x^3 - 2x\right]_2^4 \\ &= 16 + 56 - 4 = 68 \end{aligned}$$

[9] Properties of integration