Section 15.1
Integration by parts
1.5 Lectures

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MATHS 104: Mathematics for Business II

Integration by Parts

Theorem 1

(Integration by parts)

$$\int u\,dv = uv - \int v\,du$$

We have to choose u and v such that

1. u is easily differentiated.
2. v is easy to be integrated.

Most of the time, we choose u according to the rule

LIPE

where

1. L stands for “logarithmic functions“
2. I stands for “inverse functions“
3. P stands for “power functions“
4. E stands for “exponential functions“

Example 2

(Old Final Exam Question) Find $\int xe^x \, dx$

Solution: We need to use the integration by parts.

$$u = x \qquad dv = e^x \, dx$$
$$du = 1 \, dx \qquad v = e^x$$

$$\begin{aligned}\int xe^x \, dx &= uv - \int v \, du \\ &= xe^x - \int e^x \, dx \\ &= xe^x - e^x + C\end{aligned}$$

Example 3

Find $\int x^2 \ln x \, dx$

Solution: We need to use the integration by parts.

$$u = \ln x \qquad dv = x^2 \, dx$$
$$du = \frac{1}{x} \, dx \qquad v = \frac{1}{3}x^3$$

$$\begin{aligned}\int x^2 \ln x \, dx &= uv - \int v \, du \\ &= \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \frac{1}{x} \, dx \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C\end{aligned}$$

Example 4

(Old Final Exam Question) Find $\int x^2 e^x \, dx$

Solution: We need to use the integration by parts.

$$u = x^2 \qquad dv = e^x \, dx$$
$$du = 2x \, dx \qquad v = e^x$$

$$\begin{aligned}
\int x^2 e^x \, dx &= uv - \int v \, du \\
&= x^2 e^x - \int 2xe^x \, dx \\
&= x^2 e^x - 2\left(uv - \int v \, du\right) \\
&= x^2 e^x - 2\left(xe^x - \int e^x \, dx\right) \\
&= x^2 e^x - 2\left(xe^x - e^x\right) \\
&= x^2 e^x - 2xe^x + 2e^x + C
\end{aligned}$$

Example 5

(Old Final Exam Question) Find $\int \sqrt{x} \ln x^9 \, dx$

Solution: We need to use the integration by parts.

$$u = \ln x^9 \qquad dv = \sqrt{x}\, dx$$

$$du = 9\frac{1}{x}\, dx \qquad v = \frac{2}{3}x^{\frac{3}{2}}$$

$$\begin{aligned}\int \sqrt{x} \ln x^9 \, dx &= uv - \int v \, du \\ &= \frac{2}{3}x^{\frac{3}{2}} \ln x^9 - \int \frac{2}{3}x^{\frac{3}{2}} \frac{9}{x}\, dx \\ &= \frac{2}{3}x^{\frac{3}{2}} \ln x^9 - \int \frac{18}{3}x^{\frac{1}{2}}\, dx \\ &= \frac{2}{3}x^{\frac{3}{2}} \ln x^9 - 4x^{\frac{3}{2}} + C\end{aligned}$$

Example 6

(Old Final Exam Question) Find $\int_1^e \ln x \, dx$

Solution: We need to use the integration by parts.

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}\, dx \qquad v = x$$

$$\begin{aligned}
\int \ln x \, dx &= uv - \int v \, du \\
&= [x \ln x]_1^e - \int_1^e x \frac{1}{x} \, dx \\
&= [x \ln x]_1^e - \int_1^e dx \\
&= [x \ln x]_1^e - [x]_1^e \\
&= e - e + 1 = 1
\end{aligned}$$

Example 7

(Old Final Exam Question) Find $\int (e^x + x)^2 \, dx$

Solution:

$$\begin{aligned}
\int (e^x + x)^2 \, dx &= \int (e^{2x} + 2xe^x + x^2) \, dx \\
&= \frac{1}{2}e^{2x} + 2(uv - \int v \, du) + \frac{1}{3}x^3 \\
&= \frac{1}{2}e^{2x} + 2(xe^x - \int e^x \, dx) + \frac{1}{3}x^3 \\
&= \frac{1}{2}e^{2x} + 2(xe^x - e^x) + \frac{1}{3}x^3 + C
\end{aligned}$$

Example 8

Find $\int x\sqrt{3x+1}\,dx$

Solution: We need to use the integration by parts.

$$u = x \qquad dv = \sqrt{3x+1} = (3x+1)^{\frac{1}{2}}\,dx$$

$$du = 1\,dx \qquad v = \frac{2}{9}(3x+1)^{\frac{3}{2}}$$

$$\begin{aligned}\int x\sqrt{3x+1}\,dx &= uv - \int v\,du \\ &= x\frac{2}{9}(3x+1)^{\frac{3}{2}} - \int \frac{2}{9}(3x+1)^{\frac{3}{2}}\,dx \\ &= x\frac{2}{9}(3x+1)^{\frac{3}{2}} - \frac{2}{9}\int (3x+1)^{\frac{3}{2}}\,dx \\ &= x\frac{2}{9}(3x+1)^{\frac{3}{2}} - \frac{4}{45}(3x+1)^{\frac{5}{2}}\end{aligned}$$

Example 9

Find $\int (\ln x)^2 \, dx$

Solution: We need to use the integration by parts.

$$u = (\ln x)^2 \qquad dv = 1\, dx$$
$$du = 2\ln x \frac{1}{x}\, dx \qquad v = x$$

$$\begin{aligned}
\int (\ln x)^2 \, dx &= uv - \int v\, du \\
&= x(\ln x)^2 - \int 2\ln x \frac{1}{x} x \, dx \\
&= x(\ln x)^2 - 2\int \ln x \, dx \\
&= x(\ln x)^2 - 2(x\ln x - x) + C \qquad \text{See Example 6}
\end{aligned}$$

Example 10

Find $\int x^3 e^{x^2}\,dx$

Solution: We need to use the integration by parts.

$$u = x^2 \qquad dv = xe^{x^2}\,dx$$
$$du = 2x\,dx \qquad v = \frac{1}{2}e^{x^2}$$

$$\begin{aligned}\int x^3 e^{x^2}\,dx &= uv - \int v\,du \\ &= \frac{1}{2}x^2 e^{x^2} - \int xe^{x^2}\,dx \\ &= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C\end{aligned}$$