Section 3.8
Derivative of the inverse function and logarithms
3 Lecture

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MATHS 101: Calculus I

Topics

1. Inverse Functions (1 lecture).
2. Logarithms.
3. Derivative of the inverse function (1 lecture).
4. Logarithmic differentiation (1 lecture).

2- Logarithmic Function

Consider the exponential function $f(x) = a^x$.
Question: Does $f(x)$ has an inverse? Why?
Answer:

- $f^{-1}(x)$ is called and it is denoted by

$$f^{-1}(x) =$$

Note: (The fundamental equations)

1. $f(f^{-1})(x) = x$, so we have
2. $f^{-1}(f(x)) = x$, so we have

$$\underbrace{\qquad}_{\text{logarithmic form}} \quad \text{if and only if} \quad \underbrace{\qquad}_{\text{exponential form}}$$

If $a = e = 2.718281828\ldots$ (Euler number), then we simply write $\log_e$ as and it is called the

Properties of Logarithms

1. $\log_a(m \cdot n) =.$
2. $\log_a(\frac{m}{n}) =.$
3. $\log_a m^r =.$
4. $\log_a 1 =.$
5. $\log_a a =.$
6. (change of bases) $\log_a m =.$

Exercise

Use the fundamental equations to prove these six properties of the logarithms.

Example

(Expansion) Write the following expression as sum or difference of logarithms

1. $\ln(\frac{x}{wz^2}) =$
2. $\ln(\frac{x+1}{x+5})^4 =$
3. $\ln(\frac{\sqrt{x}}{(x^2)(x+3)^4}) =$

Exercise

Write each of the following expression as sum or difference of logarithms:
(1) $\log_3(\frac{5\cdot 7}{4})$ (2) $\log_2(\frac{x^5}{y^2})$ (3) $\log(\frac{x^2 z}{wy^2})$ (4) $\ln\sqrt{\frac{x+1}{x-2}}$.

Example

Write each of the following logarithm in terms of natural logarithm.

1. $\log_3 x =$
2. $\log_6 7 =$
3. $\log_2 y =$

The derivative of the inverse function

Strategy:
Goal: We want to find $\frac{d}{dx}\left(f^{-1}(x)\right)$.

Write $y = f^{-1}(x)$, we want to find y'

Geometric Interpretation *

Note that

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$

so the slope of f^{-1} is reciprocal to the slope of f. Geometrically,

Example

Let $f(x) = x^3 - 3x^2 - 1$. Find $\frac{d}{dx}\left(f(x)\right)$ and $\frac{d}{dx}\left(f^{-1}(x)\right)$ at the point $(3, -1)$

Solution:

$$\frac{d}{dx}\left(f(x)\right) =$$

$$\frac{d}{dx}\left(f(x)\right)_{(3,-1)} =$$

$$\frac{d}{dx}\left(f^{-1}(x)\right) =$$

$$\frac{d}{dx}\left(f^{-1}(x)\right)_{(3,-1)} =$$

Derivative of ln

Example

Find $\frac{d}{dx}\left(\ln x\right)$.

Solution:

$$y = \ln x$$

Exercise

Find y' if $y = \log_a x$. (Hint: Use the change of base formula to change it to ln)

Recall

The Chain Rule

Theorem

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

$$(f(g(x)))' = \textit{derivative of outer}\,(\textit{inner}) \cdot (\textit{derivative of inner})$$

Example

Find y' for each of the following:

1. $f(x) = \ln x^2 = \ln x^2 \to y' =$
2. $f(x) = \ln(2x+3) = \ln (2x+3) \to y' =$
3. $f(x) = x \ln x \to y' =$
4. $f(x) = \ln(\ln x) = \ln (\ln x) \to y' =$
5. $f(x) = \ln(\sin x) = \ln (\sin x) \to y' =$
6. $f(x) = \sin(\ln x) = \sin (\ln x) \to y' =$

Derivative using the properties of Logarithms

Example

Find the derivative of

1. $f(x) = \ln x^{2017}$

Solution: First we re–write the function in terms using the properties of the ln to get a simplified function:

$$f(x) =$$

Hence

$$f'(x) =$$

Exercise

Using the chain rule, find the derivative of the function of the previous example *without using the properties of the* ln, i.e., find $f'(x)$ for

$$f(x) = \ln(x^{2017})$$

Derivative using the properties of Logarithms

Example

Find the derivative of

1. $f(x) = \ln \sqrt[3]{\frac{x^3-1}{x^3+1}}$

Solution: First we re–write the function in terms using the properties of the ln to get a simplified function:

$$f(x) = \ln \left(\frac{x^3-1}{x^3+1}\right)^{\frac{1}{3}}$$

Continue...

We write the inner function in blue and the outer function in red and we apply the chain rule.

$$\text{derivative of outer}\,(\text{inner}) \cdot (\text{derivative of inner})$$

$$f(x) =$$
$$f'(x) =$$

Exercise

Using the chain rule, find the derivative of the function of the previous example *without using the properties of the* ln, i.e., find $f'(x)$ for

$$f(x) = \ln\left(\sqrt[3]{\frac{x^3-1}{x^3+1}}\right)$$

Example

Find $\frac{d^4y}{dx^4}$ for

$$y = 5\ln x$$

Solution: