Section 2.5
Continuity
2 Lectures

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MATHS 101: Calculus I

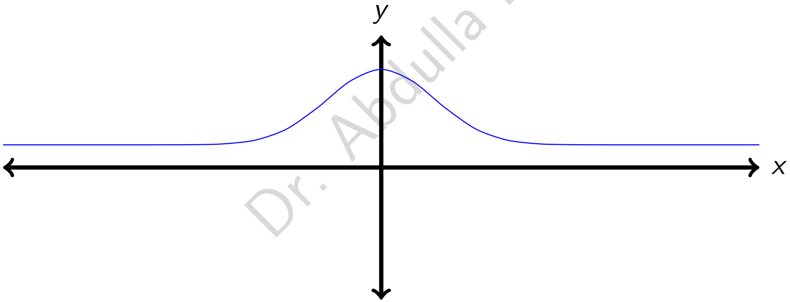
Topics:

1. Continuous functions on a point (piece–wise functions).
2. Continuous functions on an interval (other functions).
3. The intermediate value theorem (application of calculus).

Intuitive Idea

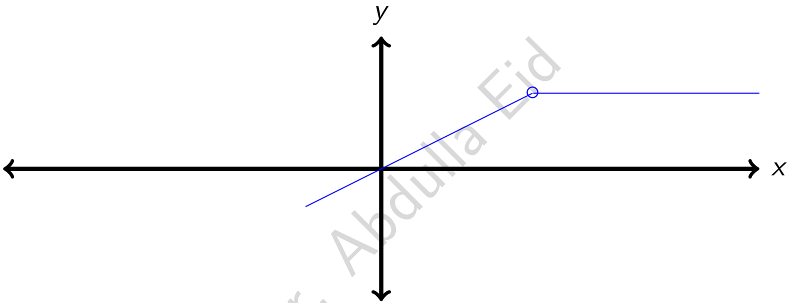
Motivational Question: What is a continuous function?

Answer: Intuitively, a function f is **continuous** function if we can sketch the graph of the function without lifting off the pencil.



"Continuous function"

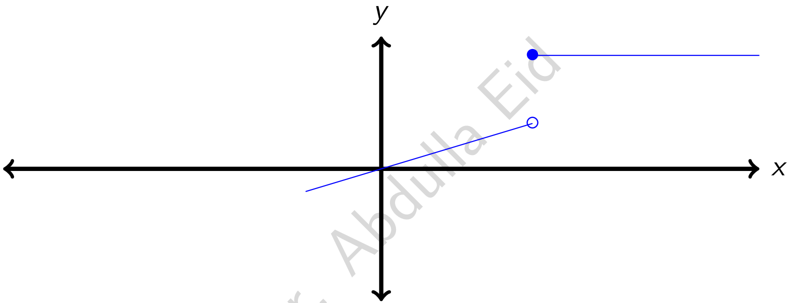
Non continuous functions



" Not Continuous function (A hole is removed)"

" Removable discontinuity"

Non continuous functions



"Not Continuous function"

"Jump discontinuity"

Continue...

To check if a function is continuous, we have two ways:

Geometry

sketch the graph of the function and check if you can trace the graph **without** lifting off the pencil
Tedious to graph a function!

Calculus

Use the limits!
easier!

Continuity using calculus

To determine whether a function is continuous at a point a using the limit, we check:

1. $\lim_{x\to a} f(x)$ exist. ($\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$). (No jumps)
2. $f(a)$ exist.
3. $\lim_{x\to a} f(x) = f(a)$. (No holes)

Piece–wise functions

Example 1

Determine whether the function is continuous at $x = 2$ or not.

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x < 2 \\ 3x - 2, & x > 2 \\ x^2, & x = 2 \end{cases}$$

Solution:

- $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 3x - 2 = 4.$
- $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-4}{x-2} = = \lim_{x\to 2^-} \frac{(x-2)(x+2)}{(x-2)} = \lim_{x\to 2^-}(x+2) = 4.$

1. $\lim_{x\to 2} f(x) = 4$, since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$.
2. $f(2) = (2)^2 = 4$

Therefore, the function is continuous at $x = 2$.

Exercise 2

Consider

$$f(x) = \begin{cases} \frac{3x+1}{x+2}, & x \neq 2 \\ 5, & x = 2 \end{cases}$$

Solution:

- $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{3x+1}{x+2} = = \frac{7}{4}$.
- $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{3x+1}{x+2} = = \frac{7}{4}$.

1. $\lim_{x\to 2} f(x) = \frac{7}{4}$, since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$.
2. $f(2) = 5$.

The function is not continuous at $x = 2$, since $\lim_{x\to 2} f(x) \neq f(2)$.

Example 3

Consider

$$f(x) = \begin{cases} \frac{x^3-8}{x-2}, & x \neq 2 \\ 12, & x = 2 \end{cases}$$

Solution:

- $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^3-8}{x-2} = \lim_{x\to 2^-} \frac{(x-2)(x^2+2x+4)}{(x-2)} = \lim_{x\to 2^-}(x^2+2x+4)12.$
- $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^3-8}{x-2} = \lim_{x\to 2^+} \frac{(x-2)(x^2+2x+4)}{(x-2)} = \lim_{x\to 2^+}(x^2+2x+4)12.$

1. $\lim_{x\to 2} f(x) = 12$, since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$.
2. $f(2) = 12$.

The function is continuous at $x = 2$, since $\lim_{x\to 2} f(x) = f(2)$.

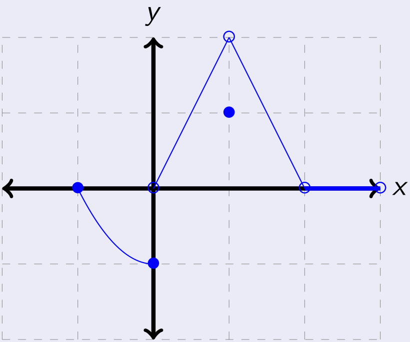
Exercise 4

Consider

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & 1.5 < x < 2 \\ 5x^2+1, & x \geq 2 \\ \frac{x^2-1}{x-1}, & x \neq 1 \\ 3, \quad x < 0 \end{cases}$$

1. Is the function continuous at $x = 2$
2. Is the function continuous at $x = 1$
3. Is the function continuous at $x = 0$

Exercise 5



1. Does $f(-1)$ exist?
2. Is the function continuous at $x = -1$
3. Is the function continuous at $x = 1$
4. Is the function continuous at $x = 2$
5. Is the function continuous at $x = 0$
6. Where the function is continuous?
7. What should be the value of $f(2)$ for the function to be continuous

Example 6

For what value of a is the function

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at $x = 3$.

Solution:

Since the function is continuous at $x = 3$, then we must have the left limit equal to the right limit.

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = f(3)$$

$$\lim_{x \to 3^+} 2ax = \lim_{x \to 3^-} x^2 - 1 = 6a$$

$$6a = 8 = 6a \to 6a = 8$$

$$a = \frac{8}{6}$$

Example 7

For what value of a and b is the function

$$f(x) = \begin{cases} 3ax, & x < 1 \\ 5x + b, & x > 1 \\ 6, & x = 1 \end{cases}$$

continuous at $x = 1$.

Solution:

Since the function is continuous at $x = 1$, then we must have the left limit equal to the right limit equal the value of the function at $x = 1$.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

$$\lim_{x \to 1^+} 3ax = \lim_{x \to 1^-} 5x + b = 6$$

$$3a = 5 + b = 6 \to 3a = 6 \quad 5 + b = 6$$

$$a = 2 \quad b = 1$$

Exercise 8

For what value(s) of a is the function

$$f(x) = \begin{cases} a^2x - 2a, & x \geq 2 \\ 12, & x < 2 \end{cases}$$

continuous at $x = 2$.

Nowhere continuous function

Exercise 9

(Challenging problem) Show whether the following function is continuous or not at any number of your choice.

$$f(x) = \begin{cases} 0, & x \text{ is rational} \\ 1, & x \text{ is irrational} \end{cases}$$

2 - Continuous functions on intervals (other functions)

Definition 10

A function is **continuous** on an interval if it is continuous at every point of the interval.

Other functions

Question: How to check if a given function is continuous on some interval?
Answer: we have a shortcut which is given in the following theorem:

Theorem 11

The functions are continuous at every point in their domain:

1. *Polynomials.* $(-\infty, \infty)$.
2. *Rational functions.* $(-\infty, \infty)$ *except where denominator =0.*
3. *Root function. inside* ≥ 0 *in case of even root.*
4. *Trigonometric functions.*
5. *Exponential functions.* $(-\infty, \infty)$.
6. *Logarithmic functions. inside* ≥ 0

In short, finding where the function is continuous is exactly the same as finding the domain of the function.

Example 12

(Zero denominator) Find the points of discontunity of $f(x) = \frac{3}{x-1}$.

Solution: Here we would have problems (undefined values) only if the denominator is equal to zero, so we need to find when the denominator is equal to zero.

$$\text{denominator} = 0 \rightarrow x - 1 = 0 \rightarrow x = 1$$

So the function is discontinuous only at $x = 1$.

Exercise 13

(Zero denominator) Find the domain of $f(x) = \frac{x^2-1}{3x^2-5x-2}$.

Solution: Similarly to the previous example, we would have problems (undefined values) only if the denominator is equal to zero,

$$\text{denominator} = 0 \rightarrow 3x^2 - 5x - 2 = 0 \rightarrow x = 2 \quad \text{or} \quad x = \frac{-1}{3}$$

So the function is discontinuous only at $x = 2$ and $x = \frac{-1}{3}$

Example 14

(Negative inside the root)the interval where the function $f(x) = \sqrt{2x-4}$ is continuous.

Solution: Here we would have problems (undefined values) only if there is a negative inside the square root, so we need to find all values that make $2x-4$ is greater than or equal to zero, so we need to solve the inequality

$$\text{inside} \geq 0 \rightarrow 2x - 4 \geq 0 \rightarrow x \geq 2$$

So the domain of f is the set of all values x such that $x \geq 2$, i.e., $[2, \infty)$

Exercise 15

(Negative inside the root and zero in the denominator) Find the interval(s) where the following function is continuous $f(x) = \frac{3}{\sqrt{x-4}}$.

Solution: Here we would have two problems (undefined values) only if there is a negative inside the square root or zero in the denominator, so we need to find all values that make $x - 4$ is is equal to zero and we exclude them. Then we find all the values that make $x - 4$ non–negative, so we need to solve the first

$$\text{denominator} = 0 \qquad \text{and} \qquad \text{inside} \geq 0$$

$$x - 4 = 0 \text{ and } x - 4 \geq 0$$

So the domain of f is the set of all values x such that $x \geq 4$ and $x \neq 4$, i.e., $(4, \infty)$

Properties of continuous functions

Theorem 16

If f and g are two continuous functions on some interval, then so $f \pm g$, $f \cdot g$, , $\frac{f}{g}(g \neq 0)$,f^n, $\sqrt[n]{f}$ (based on the domain).

Exercise 17

If f is continuous function at a and g is continuous function at $b = f(a)$, then the composite $g \circ f$ is continuous function at a.

(Hint: Compute $\lim_{x \to a}(g \circ f)(x)$)

3 - Intermediate Value Theorem (Application of Calculus)

Here we give an application of calculus to finding the location of a root to an equation.

Definition 18

A number c is called a **root** (**zero**) for a function f if

$$f(c) = 0$$

Theorem 19

Let f be a continuous function on an interval $[a, b]$ such that $f(a)$ and $f(b)$ have different signs , then there exist a root $c \in (a, b)$ such that $f(c) = 0$.

Example 20

Show there exists a root for $x^3 - x - 1 = 0$ between 1 and 2.

Solution: Note that the function is continuous (polynomial) and we have

$$f(1) = -1 < 0 \quad f(2) = 5 > 0$$

Therefore, by the IVT, there must be a root $c \in (1, 2)$ such that $f(c) = 0$. The IVT does not tell us how to find that root.

Exercise 21

Show there exists a root for $x^3 - 3x - 1 = 0$.

Solution: Note that the function is continuous (polynomial). Here we do not have an interval, so we need to find a suitable interval (two end–points with different sign). One choice is 0, so we have $f(0) = -1 < 0$. Now we look for some other number with a positive value, for example, $f(2) = 1 > 0$. Therefore, by the IVT, there must be a root $c \in (0, 2)$ such that $f(c) = 0$. The IVT does not tell us how to find that root.

Example 22

Show there exist a number $c \in (0,1)$ such that $\sqrt[3]{c} = 1 - c$.

Solution: The problem can be translate as to find a number $c \in (0,1)$ such that $\sqrt[3]{c} - 1 + c = 0$, i.e., we need to show that there is a root for the equation $\sqrt[3]{x} - 1 + x = 0$. Note that the function is continuous. We have $f(0) = -1 < 0$ and $f(1) = 1 > 0$ Therefore, by the IVT, there must be a root $c \in (0,1)$ such that $f(c) = 0$, i.e., we have $\sqrt[3]{c} = 1 - c$.

Exercise 23

(Challenging Problem) **The fixed point theorem** Suppose f is a continuous function on $[0,1]$ such that $0 \leq f(x) \leq 1$. Show there exist $c \in (0,1)$ such that $f(c) = c$.

(Hint: Apply IVT to $g(x) = f(x) - x$).