Section 2.6
Limits at infinity and infinite limits
2 Lectures

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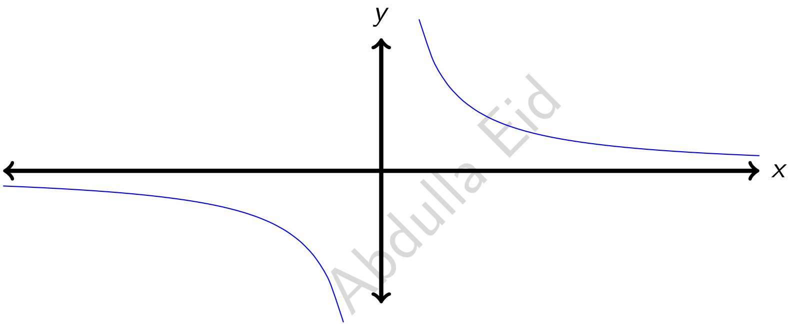
MATHS 101: Calculus I

1. Finite limits as $x \to \pm\infty$.
2. Horizontal Asympotes.
3. Infinite limits.
4. Vertical Asymptotes.

Motivation Example

Consider the function $f(x) = \frac{1}{x}$. The graph of the function is



Question: What happen if x is sufficiently large number (i.e., x approaches ∞) ? In other words, what is $\lim_{x\to\infty} \frac{1}{x}$?
From the graph we can easily see that

$$\boxed{\lim_{x\to\infty} \frac{1}{x} = 0} \quad \text{and} \quad \boxed{\lim_{x\to-\infty} \frac{1}{x} = 0}$$

Continue...

Arithmetic at infinity:

1. $\infty + \infty = \infty$.
2. $k \cdot \infty = \infty$ $(k > 0)$.
3. $k \cdot \infty = -\infty$ $(k < 0)$.
4. $\frac{1}{\pm\infty} = 0$.
5. $\frac{\infty}{\infty} = ?$. (Calculus 1).
6. $\infty - \infty =?$, $0 \cdot \infty =?$, $1^{\infty} =?$. (Calculus 2)

Finding the limit of a rational function

To find the limit $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)}$, we have

1. Substitute directly by $x = \pm\infty$ in $\frac{f(x)}{g(x)}$. If you get a real number or $\pm\infty$, then that is the limit.
2. If you get undefined values such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we take the highest power of x in the numerator and the highest power of x in the denominator as common factor and we proceed.

Example 1

Find

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

Solution: Direct substitution gives

$$\frac{3(\infty)^2 - (\infty) - 2}{5(\infty)^2 + 4(\infty) + 1} \qquad \text{undefined!}$$

$$\begin{aligned}\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} &= \lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} \\ &= \lim_{x\to\infty} \frac{x^2\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{x^2\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)} \\ &= \lim_{x\to\infty} \frac{\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)} \\ &= \frac{(3 - 0 - 0)}{(5 + 0 + 0)} = \frac{3}{5}\end{aligned}$$

Exercise 2

Find

$$\lim_{x\to\infty} \frac{5x^2 - 2x + 1}{9x^2}$$

Example 3

Find

$$\lim_{x\to\infty} \frac{3x+7}{x^2-2}$$

Solution: Direct substitution gives

$$\frac{3\infty+7}{(\infty)^2-2} = \frac{\infty}{\infty} \quad \text{undefined!}$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x+7}{x^2-2} &= \lim_{x\to\infty} \frac{3x+7}{x^2-2} \\
&= \lim_{x\to\infty} \frac{x\left(3+\frac{7}{x}\right)}{x^2\left(1-\frac{2}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{\left(3+\frac{7}{x}\right)}{x\left(1-\frac{2}{x^2}\right)} \\
&= \frac{(3+0)}{\infty(1-0)} = 0
\end{aligned}$$

Example 4

Find

$$\lim_{x\to\infty} \frac{x^3-8}{2x^2+1}$$

Solution: Direct substitution gives

$$\frac{(\infty)^3-8}{2(\infty)^2+1} = \frac{\infty}{\infty} \quad \text{undefined!}$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3-8}{2x^2+1} &= \lim_{x\to\infty} \frac{x^3-8}{2x^2+1} \\
&= \lim_{x\to\infty} \frac{x^3\left(1-\frac{8}{x^3}\right)}{x^2\left(2+\frac{1}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{x\left(1-\frac{8}{x^3}\right)}{\left(2+\frac{1}{x^2}\right)} \\
&= \infty
\end{aligned}$$

Exercise 5

Find

$$\lim_{x\to\infty} \frac{5x^3 - 2x + 1}{9x^8 + 8}$$

Example 6

Find

$$\lim_{x\to\infty} \frac{\sqrt{3x^2+1}}{3x-5}$$

Solution: Direct substitution gives

$$\lim_{x\to\infty} \frac{\sqrt{3(\infty)^2+1}}{3(\infty)-5} = \frac{\infty}{\infty} \quad \text{undefined!}$$

$$\lim_{x\to\infty} \frac{\sqrt{3x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{3x^2+1}}{3x-5}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2\left(3+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{\left(3+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{|x|\sqrt{\left(3+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)} = \lim_{x\to\infty} \frac{x\sqrt{\left(3+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)}$$

$$= \lim \sqrt{\left(3+\frac{1}{x^2}\right)} = \sqrt{3}$$

Exercise 7

Find

$$\lim_{x \to -\infty} \frac{\sqrt{3x^2+1}}{3x-5}$$

Solution: Direct substitution gives

$$\lim_{x \to -\infty} \frac{\sqrt{3(-\infty)^2+1}}{3(-\infty)-5} = \frac{\infty}{-\infty} \quad \text{undefined!}$$

$$\lim_{x \to -\infty} \frac{\sqrt{3x^2+1}}{3x-5} = \lim_{x \to \infty} \frac{\sqrt{3x^2+1}}{3x-5}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{x^2\left(3+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)} = \lim_{x \to -\infty} \frac{\sqrt{x^2}\sqrt{\left(3+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)}$$

$$= \lim_{x \to -\infty} \frac{|x|\sqrt{\left(3+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)} = \lim_{x \to -\infty} \frac{-x\sqrt{\left(3+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)}$$

$$= \lim \frac{-\sqrt{\left(3+\frac{1}{x^2}\right)}}{\ldots} = \frac{-\sqrt{3}}{\ldots}$$

Multiplying by the conjugate

Example 8

Find

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$$

Solution: Direct substitution gives

$$\left(\sqrt{(\infty)^2+1}-x\right)=\infty-\infty \quad \text{undefined!}$$

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\cdot\frac{\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}+x\right)}$$

$$=\lim_{x\to\infty}\frac{x^2+1-x^2}{\left(\sqrt{x^2+1}+x\right)}=\lim_{x\to\infty}\frac{1}{\left(\sqrt{x^2+1}+x\right)}$$

$$=0$$

Exercise 9

Find

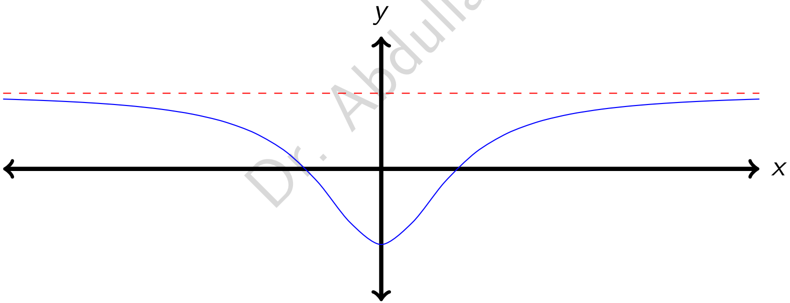
$$\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)$$

2 - Horizontal Asymptotes

Motivational Example: Consider the function $f(x) = \frac{x^2-1}{x^2+1}$. Then we have

$$\lim_{x\to\infty} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = 1$$

In this case, the line $y = 1$ is called a **horizontal asymptote**.



Definition 10

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{and} \qquad \lim_{x\to-\infty} f(x) = L$$

Example 11

Find the horizontal asymptote of the function

$$f(x) = \frac{x-9}{\sqrt{4x^2+3x+2}}$$

we need to find both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

$$\lim_{x\to\infty} \frac{x-9}{\sqrt{4x^2+3x+2}} = \lim_{x\to\infty} \frac{x-9}{\sqrt{3x^2+3x+1}}$$

$$= \lim_{x\to\infty} \frac{x\left(1-\frac{9}{x}\right)}{\sqrt{x^2\left(4+\frac{3}{x}+\frac{2}{x^2}\right)}} = \lim_{x\to\infty} \frac{x\left(1-\frac{9}{x}\right)}{\sqrt{x^2}\sqrt{\left(4+\frac{3}{x}+\frac{2}{x^2}\right)}}$$

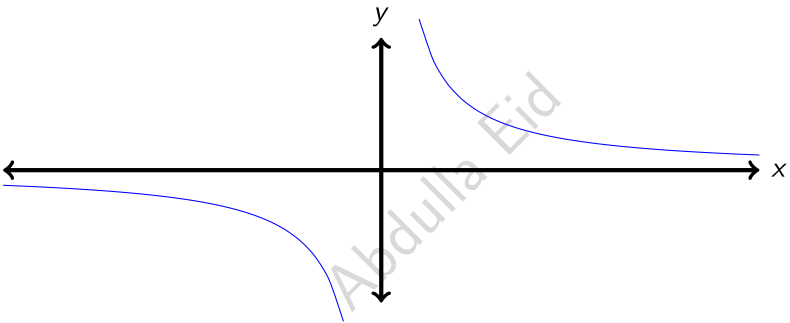
$$= \lim_{x\to\infty} \frac{x\left(1-\frac{9}{x}\right)}{|x|\sqrt{\left(4+\frac{3}{x}+\frac{2}{x^2}\right)}} = \lim_{x\to\infty} \frac{x\left(1-\frac{9}{x}\right)}{x\sqrt{\left(4+\frac{3}{x}+\frac{2}{x^2}\right)}}$$

$$= \lim_{x\to\infty} \frac{\left(1-\frac{9}{x}\right)}{\sqrt{\left(4+\frac{3}{x}+\frac{2}{x^2}\right)}} = \frac{1}{2}$$

Hence $\boxed{y = \frac{1}{2}}$ is a horizontal asymptote. Now we compute $\lim_{x\to-\infty} f(x)$ to get $\lim_{x\to-\infty} f(x) = \frac{-1}{2}$ and so we have $\boxed{y = \frac{-1}{2}}$ is also a horizontal asymptote.

Motivation Example

Consider the function $f(x) = \frac{1}{x}$. The graph of the function is



Question: What is $\lim_{x \to 0^+} \frac{1}{x}$ and $\lim_{x \to 0^-} \frac{1}{x}$? From the graph we can easily see that

$$\boxed{\lim_{x \to 0^+} \frac{1}{x} = \infty} \quad \left(\frac{1}{0^+} = \infty\right) \quad \text{and} \quad \boxed{\lim_{x \to 0^-} \frac{1}{x} = \infty} \left(\frac{1}{0^+} = \infty\right)$$

Example 12

Find

$$\lim_{x\to 1^+} \frac{3}{x-1}$$

Solution: Direct substitution gives

$$\frac{3}{0} \quad \text{undefined!}$$

So we need to find whether it is 0^+ or 0^-.

$$\lim_{x\to 1^+} \frac{3}{x-1} = \frac{3}{0^+}$$
$$= \infty$$

Exercise 13

Find

$$\lim_{x\to 1^-} \frac{3}{x-1}$$

Solution: Direct substitution gives

$$\frac{3}{0} \quad \text{undefined!}$$

So we need to find whether it is 0^+ or 0^-.

$$\lim_{x\to 1^-} \frac{3}{x-1} = \frac{3}{0^-}$$
$$= -\infty$$

Example 14

Find

$$\lim_{x\to -1^+} \frac{-2}{x+1}$$

Solution: Direct substitution gives

$$\frac{-2}{0} \quad \text{undefined!}$$

So we need to find whether it is 0^+ or 0^-.

$$\lim_{x\to -1^+} \frac{-2}{x+1} = \frac{-2}{0^+}$$
$$= -2 \cdot \infty$$
$$= -\infty$$

Exercise 15

Find

$$\lim_{x\to 2^+} \frac{3}{2-x}$$

Solution: Direct substitution gives

$$\frac{3}{0} \quad \text{undefined!}$$

So we need to find whether it is 0^+ or 0^-.

$$\lim_{x\to 2^+} \frac{3}{2-x} = \frac{3}{0^-}$$
$$= -\infty$$

Example 16

Find

$$\lim_{x \to 4^-} \frac{2x}{x^2 - 16}$$

Solution: Direct substitution gives

$$\frac{8}{0} \quad \text{undefined!}$$

So we need to find whether it is 0^+ or 0^-.

$$\lim_{x \to 4^-} \frac{2x}{x^2 - 16} = \frac{8}{0^-}$$
$$= -\infty$$

Example 17

Find

$$\lim_{x \to 2^+} \frac{x-2}{x^2-4x+4}$$

Solution: Direct substitution gives

$$\frac{0}{0} \quad \text{undefined!}$$

So we need to factor first using the methods of Section 2.2.

$$\lim_{x \to 2^+} \frac{x-2}{x^2-4} = \lim_{x \to 2^+} \frac{(x-2)}{(x-2)(x-2)} = \lim_{x \to 2^+} \frac{1}{x-2}$$

So we need to find whether it is 0^+ or 0^-.

$$\lim_{x \to 2^+} \frac{1}{x-2} = \frac{1}{0^+}$$
$$= \infty$$

Exercise 18

Find

$$\lim_{x\to 3} \frac{3x}{x^2-9}$$

Solution: Direct substitution gives

$$\frac{9}{0} \quad \text{undefined!}$$

So we need to find whether it is 0^+ or 0^- and for that we find the right and the left limits.

$$\lim_{x\to 3^+} \frac{3x}{x^2-9} = \frac{9}{0^+} = \infty$$

$$\lim_{x\to 3^-} \frac{3x}{x^2-9} = \frac{9}{0^-} = -\infty$$

Since $\lim_{x\to 3^+} \frac{3x}{x^2-9} \neq \lim_{x\to 3^-} \frac{3x}{x^2-9}$, we have

$$\lim_{x\to 3} \frac{3x}{x^2-9} \quad \text{Does Not Exist}$$

Example 19

Find

$$\lim_{x\to 1} \frac{x^2-1}{(x-1)^2}$$

Solution: Direct substitution gives $\frac{0}{0}$ undefined! So we need to factor first using the methods of Section 2.2.

$$\lim_{x\to 1} \frac{x^2-1}{(x-1)^2} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x-1)} = \lim_{x\to 1} \frac{x+1}{x-1}$$

So we need to find whether it is 0^+ or 0^- and for that we find the right and the left limits.

$$\lim_{x\to 1^+} \frac{x+1}{x-1} = \frac{2}{0^+} = \infty \qquad \lim_{x\to 1^-} \frac{x+1}{x-1} = \frac{2}{0^-} = -\infty$$

Since $\lim_{x\to 1^+} \frac{x^2-1}{(x-1)^2} \neq \lim_{x\to 1^-} \frac{x^2-1}{(x-1)^2}$, we have

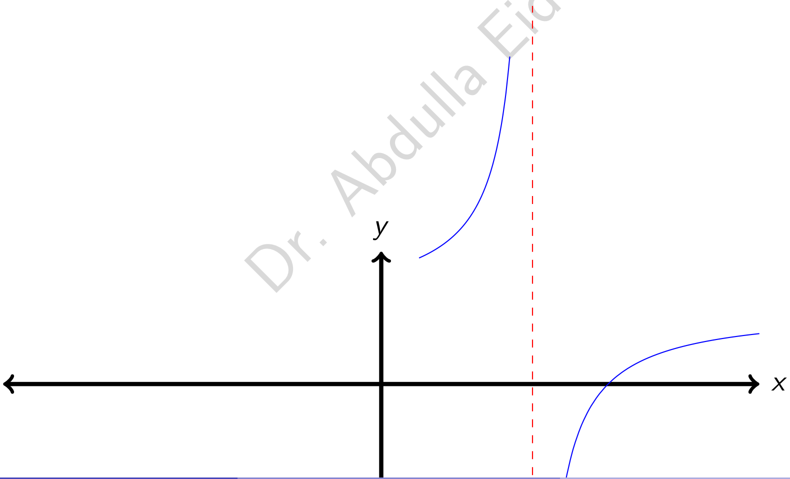
$$\lim_{x\to 1} \frac{x^2-1}{(x-1)^2}$$ Does Not Exist

4 - Vertical Asymptotes

Motivational Example: Consider the function $f(x) = \frac{x+3}{x-2}$. Then we have

$$\lim_{x \to 2^+} f(x) = \infty \qquad \text{and} \qquad \lim_{x \to 2^-} f(x) = -\infty$$

In this case, the line $x = 2$ is called a **vertical asymptote**.



Definition 20

The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty$$

To find the vertical asymptote for a rational function, we need to cancel any common factor first and we find where the denominator is zero.

Example 21

Find the vertical asymptote of the function

$$f(x) = \frac{-8}{x^2 - 4}$$

$x^2 - 4 = 0 \to x = -2, x = 2.$ Since none of these is a zero for the numerator, then both are vertical asymptote.

Exercise 22

Find the vertical asymptote of the function

$$f(x) = \frac{x^2 - 4x + 3}{x^2 - 1}$$

$\frac{x^2-4x+3}{x^2-1} \rightarrow \frac{(x-1)(x-3)}{(x-1)(x+1)} \rightarrow \frac{(x-3)}{(x+1)} \rightarrow x = -1$. So only $\boxed{x = -1}$ is a vertical asymptote.