Section 3.11
Linear Approximation and Differentials
1.5 Lectures

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MATHS 101: Calculus I

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In this section, we will study:

1 - Linear Approximation

Idea: Given a function f, a number a, and a number x very close to a such that

$$f(a), f'(a) \text{ can be easily computed}$$
$$f(x) \text{ is difficult to compute}$$

Goal: We use $f(a), f'(a)$ to approximate the value $f(x)$.

Example 1

Consider $f(x) = \sqrt{x}$, $a = 16$, $x = 16.01$. Then

$$f(16) = \sqrt{16} = 4 \quad \text{easy}$$
$$f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8} \quad \text{easy}$$
$$f(16.01) = \sqrt{16.01} = ?? \quad \text{difficulut}$$

Linear Approximation

Given a function f, a number a, and a number x very close to a, we have

$$\begin{aligned} f(x) &\sim \text{Tangent line at } a \\ &\sim y_1 + m(x - x_1) \\ &\sim f(a) + f'(a)(x - a) \end{aligned}$$

Definition 2

The **linear approximation** of f at a is given by

$$f(a) + f'(a)(x - a)$$

Example 3

Find the linear approximation of $f(x) = \sqrt{x}$ at $a = 16$ and then use it to approximate $\sqrt{16.01}$.

Solution: We need to find $f'(x) = \frac{1}{2\sqrt{x}}$. Then we have the linear approximation is given by

$$\begin{aligned} f(x) &\sim f(a) + f'(a)(x - a) \\ &\sim \sqrt{16} + \frac{1}{2\sqrt{16}}(x - 16) \\ &\sim 4 + \frac{1}{8}(x - 16) = 4 + \frac{1}{8}x - 2 \end{aligned}$$

$$\sim 2 + \frac{1}{8}x$$

$$\sqrt{16.01} \sim 2 + \frac{1}{8}(16.01) \sim 4.00125$$

Exercise 4

Compare the answer above with the one you would get if you use a

Example 5

Find the linear approximation of $f(x) = \sqrt[3]{8-x}$ at $a = 0$.

Solution: We need to find $f'(x) = \frac{-1}{3}(8-x)^{\frac{2}{3}}$. Then we have the linear approximation is given by

$$\begin{aligned} f(x) = \sqrt[3]{8-x} &\sim f(a) + f'(a)(x-a) \\ &\sim \sqrt[3]{8-0} + \frac{-1}{3}(16-0)^{\frac{2}{3}}(x-0) \\ &\sim 2 - \frac{4}{3}(x) \\ &\sim 2 - \frac{4}{3}x \end{aligned}$$

Exercise 6

Find the linear approximation of $f(x) = \ln(x+1)$ at $a = 0$.

Example 7

Use linear approximation to approximate the value of $\sin 0.03$.

Solution: Here we know that $x = 0.03$, we need to find the function f and a value a near x that we can compute $f(a), f'(a)$ easily.
Let $f(x) = \sin x$ ($f'(x) = \cos x$) and $a = 0$

$$\begin{aligned} f(x) = \sin x &\sim f(a) + f'(a)(x - a) \\ &\sim \sin 0 + \cos 0(x - 0) = 0 + 1(x) \\ \sin x &\sim x \\ \sin 0.03 &\sim 0.03 \end{aligned}$$

Exercise 8

Compare the answer above with the one you would get if you use a calculator or a computer.

Exercise 9

Use linear approximation to approximate the value of $\ln 1.03$. (Hint: Use Exercise 6

Example 10

Given $f(1) = 3$ and $f'(x) = 2x + e^{x-1}$ for all x. Use linear approximation to estimate $f(0.9)$ and $g(1.01)$.

Exercise 11

Given $g(2) = -6$ and $g'(x) = \sqrt{x^2 + 7}$ for all x. Use linear approximation to estimate $g(2.05)$ and $g(1.95)$.

2 - Differentials

Definition 12

Let $y = f(x)$, then the differential dy is given by

$$dy = f'(x)dx$$

Geometric Interpretation:
What is dx and dy?

Example 13

Find the differential of each of the following functions:

1. $y = 1 + 2x^3 \rightarrow dy = 6x^2\,dx$
2. $x^2 + 4y^2 = 5 \rightarrow 2x\,dx + 8y\,dy = 0 \rightarrow dy = \frac{-x}{4y}\,dx$

Exercise 14

Find the differential of each of the following functions:

1. $y = e^x + 4.$
2. $y = \cos x + \sin x.$
3. $\tan y = e^x$

Example 15

Find dx of each of the following functions:

1. $u = 3 - 4x^2 \rightarrow du = -8x\,dx \rightarrow dx = \frac{1}{-8x}\,du.$
2. $u = \sin^{-1} x \rightarrow du = \frac{1}{\sqrt{1-x^2}}\,dx \rightarrow dx = \sqrt{1-x^2}\,du.$

Exercise 16

Find dx of each of the following functions:

1. $u = ax + b.$
2. $u = 1 - \cos^2 x$

Chain Rule, Second form

Example 17

Let $y = \sin x + e^x$ and $x = t^2 + 4t$. Find $\frac{dy}{dt}$ at $t = 1$?

Solution: Since $t = 1$, we have $x = (1)^2 + 4(1) = 5$.

$$
\begin{aligned}
\frac{dy}{dt} &= \frac{(\cos x + e^x)dx}{dt} \\
&= \frac{(\cos x + e^x)(2t+4)dt}{dt} \\
&= (\cos x + e^x)(2t+4) \\
\frac{dy}{dt}_{|t=1} &= (\cos 5 + e^5)(2(1)+4)
\end{aligned}
$$

Chain Rule, Second form

Exercise 18

Let $r = \frac{2}{q} + 10q$ and $q = 7 + \frac{12}{t}$. Find $\frac{dr}{dt}$ at $t = 3$?

Solution: Since $t = 3$, we have $q = 11$.

$$\frac{dr}{dt} = \frac{(-\frac{2}{q^2} + 10)dq}{dt}$$

$$= \frac{(-\frac{2}{q^2} + 10)(-\frac{12}{t^2})dt}{dt}$$

$$= \left(-\frac{2}{q^2} + 10\right)\left(-\frac{12}{t^2}\right)$$

$$\frac{dy}{dt}_{|t=3} =$$