Section 3.3
Product and Quotient rules
1 Lecture

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MATHS 101: Calculus I

Motivation

Goal: We want to derive rules to find the derivative of product $f(x)g(x)$ and quotient $\frac{f(x)}{g(x)}$ of two functions.

Example 1

We want to find (in a general way) the derivative of the functions:

- $f(x) = (3x+1)(5x+2)$.
- $f(x) = xe^x$.
- $f(x) = \frac{3x+1}{x^3+2x+1}$.
- $f(x) = \frac{x+1}{e^x+6x-3}$.

The Product Rule (Leibniz Rule)

Theorem 2

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

$$(f(x)g(x))' = (\textit{derivative of first})\,(\textit{second}) + (\textit{first})(\textit{derivative of second})$$

Before we prove this theorem, recall that the definition of the derivative is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \text{ and } g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

Proof: Let $F(x) = f(x)g(x)$. Then,

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \end{aligned}$$

We will use a "trick" by adding and subtracting $f(x)g(x+h)$ in the middle of the numerator.

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{[f(x+h) - f(x)]g(x+h) + f(x)[g(x+h) - g(x)]}{h} \end{aligned}$$

Continue...

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{[f(x+h)-f(x)]g(x+h)+f(x)[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)-f(x)]g(x+h)}{h} + \frac{f(x)[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)-f(x)]g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(x)g(x)+f(x)g'(x)
\end{aligned}
$$

Example 3

Find the derivative of each of the following:

1. $F(x) = (x^2 + 5x - 6)(6x^2 - 5x + 6)$
2. $F(x) = 2(\sqrt{x} + 5x - 3)(\sqrt[4]{x} - 4\sqrt{x})$

Solution: (1)

$$\begin{aligned} F'(x) &= (\text{derivative of first})\,(\text{second}) + (\text{first})(\text{derivative of second}) \\ &= (2x+5)(6x^2 - 5x + 6) + (x^2 + 5x - 6)(12x - 5) \end{aligned}$$

(2)

$$\begin{aligned} F'(x) &= (\text{derivative of first})\,(\text{second}) + (\text{first})(\text{derivative of second}) \\ &= 2\left[\left(\frac{1}{2\sqrt{x}} + 5\right)(\sqrt[4]{x} - 4\sqrt{x}) + (\sqrt{x} + 5x - 3)\left(\frac{1}{4}x^{\frac{-3}{4}} - \frac{4}{2\sqrt{x}}\right)\right] \end{aligned}$$

Product rule for 3 functions or more

$$(fg)' = f'g + fg'$$

$$(fgh)' = f'gh + fg'h + fgh'$$

Example 4

Find the derivative of each of the following:

1. $F(x) = (x-1)(x-2)(x^2-4)$

Solution:

$$F'(x) = (1)(x-2)(x^2-4) + (x-1)(1)(x^2-4) + (x-1)(x-2)(2x)$$

The Quotient Rule

Theorem 5

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

$$\frac{(\textit{denominator})(\textit{derivative of numerator}) - (\textit{numerator})(\textit{derivative of denomi}}{(\textit{denominator})^2}$$

To prove this theorem, we will use the product rule.

Proof: Let $F(x) = \frac{f(x)}{g(x)}$. We want to find $F'(x)$. For that we apply the product rule to

$$F(x)g(x) = f(x)$$

$$\begin{aligned}
(\text{derivative of first})(\text{second}) + (\text{first})(\text{derivative of second}) &= f'(x) \\
F'(x)g(x) + F(x)g'(x) &= f'(x) \\
F'(x)g(x) &= f'(x) - F(x)g'(x) \\
F'(x) &= \frac{f'(x) - F(x)g'(x)}{g(x)} \\
F'(x) &= \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)} \\
F'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}
\end{aligned}$$

Example 6

Find the derivative of each of the following:

1. $F(x) = \frac{2}{5x+1}$
2. $F(x) = \frac{1-x}{1-x^3}$

Solution: (1)

$$F'(x) = \frac{(\text{denominator})(\text{derivative of numerator}) - (\text{numerator})(\text{derivative of}}{(\text{denominator})^2}$$

$$= \frac{(5x+1)(0) - (2)(5)}{(5x+1)^2}$$

$$= \frac{-10}{(5x+1)^2}$$

Continue...

Recall we want to find the derivative of $F(x) = \frac{1-x}{1-x^3}$.

$$F'(x) = \frac{(\text{denominator})(\text{derivative of numerator}) - (\text{numerator})(\text{derivative of}}{(\text{denominator})^2}$$

$$= \frac{(1-x^3)(-1) - (1-x)(-3x^2)}{(1-x^3)^2}$$

$$= \frac{-1 + x^3 + 3x^2 - 3x^3}{(1-x^3)^2}$$

$$= \frac{-1 + 3x^2 - 2x^3}{(1-x^3)^2}$$

Exercise 7

Find the derivative of the following functions:

1. $f(x) = \frac{e^x}{x}$
2. $f(x) = \frac{ax+b}{cx+d}$