Section 3.3
Differentiability of case–defined functions and higher derivative
1 Lecture

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MATHS 101: Calculus I

1 - Derivative of case–defined functions

Example 1

Show that the function

$$f(x) = \begin{cases} 3x^2 + 2x + 1, & x > 0 \\ e^{2x}, & x \leq 0 \end{cases}$$

is differentiable at $x = 0$.

Solution: We find

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 1}{h}$$

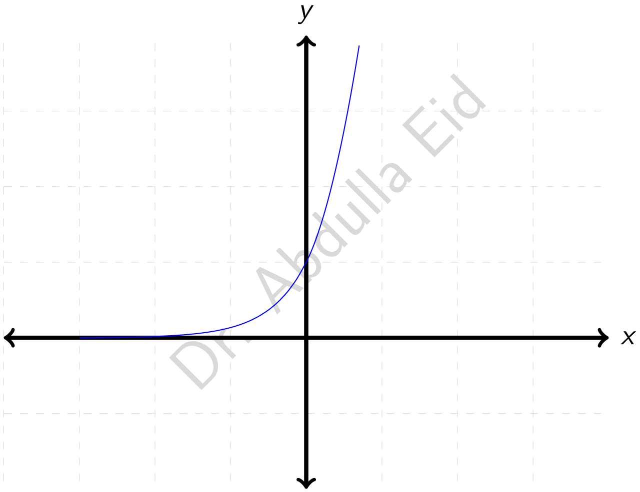
We need to compute the left and right limit.

$$\lim_{h \to 0^+} \frac{f(h) - 1}{h} = f(0^+) = 2 \qquad \lim_{h \to 0^-} \frac{f(h) - 1}{h} = f'(0^-) = 2$$

so we have

$$f'(0) = 2$$

The graph of the function



Exercise 2

Show that the function

$$f(x) = \begin{cases} -x, & x < 0 \\ \frac{x^2}{x+1}, & x \geq 0 \end{cases}$$

is differentiable at $x = 0$.

Solution: We find

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 1}{h}$$

We need to compute the left and right limit.

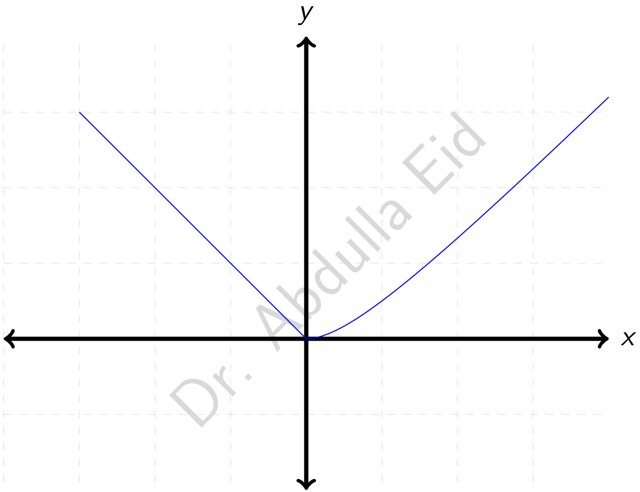
$$f(0^-) = -1$$

$$f_2(x) = \frac{(x+1)(2x) - x^2(1)}{(x+1)^2}$$

$$f(0^+) = 0$$

so we have

$$f'(0) = \text{ Does not exist}$$


y
x

Example 3

For which value(s) is the function defined by

$$f(x) = \begin{cases} ax+b, & x<0 \\ x-x^2, & x\geq 0 \end{cases}$$

differentiable at $x=0$?

Solution: We find

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$$

We need to compute the left and right limit and we make them equal.

$$f(0^-) = a \qquad\qquad f(0^+) = 1$$

so we have $a=1$.Now since the function is continuous, then we must have the right limit equal the left limit and so we have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) \to b = 0$$

Exercise 4

For which value(s) is the function defined by

$$f(x) = \begin{cases} ax + b, & x < 1 \\ x - x^6, & x \geq 1 \end{cases}$$

differentiable at $x = 1$?

2 - Higher derivatives

Let $y = f(x)$

First Derivative	y'	$\frac{dy}{dx}$	$f'(x)$	$\frac{d}{dx}(f(x))$
Second Derivative	y''	$\frac{d^2y}{dx^2}$	$f''(x)$	$\frac{d^2}{dx^2}(f(x))$
Third Derivative	y'''	$\frac{d^3y}{dx^3}$	$f'''(x)$	$\frac{d^3}{dx^3}(f(x))$
Fourth Derivative	$y^{(4)}$	$\frac{d^4y}{dx^4}$	$f^{(4)}(x)$	$\frac{d^4}{dx^4}(f(x))$
nth Derivative	$y^{(n)}$	$\frac{d^ny}{dx^n}$	$f^{(n)}(x)$	$\frac{d^n}{dx^n}(f(x))$

Example 5

Find $\frac{d^4y}{dx^4}$ for

$$y = e^{3x} + x^3 + \sqrt[3]{333}$$

Solution:

$$\begin{aligned} y' &= 3e^{3x} + 3x^2 \\ y'' &= 9e^{3x} + 6x \\ y''' &= 27e^{3x} + 6 \\ &= 81e^{3x} \end{aligned}$$

Exercise 6

Find $\frac{d^3y}{dx^3}$ for

$$y = x^4e^x$$

Solution:

$$\begin{aligned}
y' &= 4x^3e^x + x^4e^x \\
y'' &= (12x^2e^x + 4x^3e^x) + (4x^3e^x + x^4e^x) \\
&= 12x^2e^x + 8x^3e^x + x^4e^x \\
y''' &= (24xe^x + 12x^2e^x) + (24x^2e^x + 8x^3e^x) + (4x^3e^x + x^4e^x) \\
&= 24xe^x + 36x^2e^x + 12x^3e^x + x^4e^x
\end{aligned}$$