Section 3.5
Derivative of Trigonometric Functions
2 Lectures

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MATHS 101: Calculus I

Topics:

1. Review of the trigonometric functions (Pre–Calculus).
2. Limits involving trigonometric functions.
3. Derivative of the basic trigonometric functions.
4. Derivative of the functions that involve trigonometric functions.

3 - Derivative of the trigonometric functions

Theorem 1

$$\frac{d}{dx}(\sin x) = \cos x$$

Proof: Let $f(x) = \sin x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x\sin h}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0}\frac{\cos x\sin h}{h}$$

$$f'(x) = \sin x\cdot\lim_{h\to 0}\frac{(\cos h - 1)}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h}$$

$$= \sin x\cdot 0 + \cos x\cdot 1 = \cos x$$

Homework

Exercise 2

$$\frac{d}{dx}(\cos x) = \sin x$$

Theorem 3

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Proof: Let $f(x) = \tan x = \frac{\sin x}{\cos x}$

$$f'(x) = \frac{(\text{denominator})(\text{derivative of numerator}) - (\text{numerator})(\text{derivative of}}{(\text{denominator})^2}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

Homework

Exercise 4

$$\frac{d}{dx}(\cot x) = \csc^2 x$$

Theorem 5

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

Proof: Let $f(x) = \tan x = \frac{1}{\cos x}$

$$f'(x) = \frac{(\text{denominator})(\text{derivative of numerator}) - (\text{numerator})(\text{derivative of}}{(\text{denominator})^2}$$

$$= \frac{\cos x \cdot 0 - 1 \cdot (-\sin x)}{(\cos x)^2}$$

$$= \frac{1 \cdot \sin x}{\cos^2 x} = \frac{1 \cdot \sin x}{\cos x \cdot \cos x}$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$= \sec x \tan x$$

Homework

Exercise 6

$$\frac{d}{dx}\left(\csc x\right) = -\csc x \cot x$$

Summary

(Derivative of the Trigonometic function)

$(\sin x)' = \cos x$ $\quad$ $(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$ $\quad$ $(\cot x)' = -\csc x$

$(\sec x)' = \sec x \tan x$ $\quad$ $(\csc x)' = -\csc x \cot x$

4 - Derivative of the functions that involve the trigonometric functions

Example 7

Differentiate $y = x^2 - \cos x$.

Solution:

$$y' = 2x + \sin x$$

Exercise 8

Find y' for $y = x^5 \sin x$.

Solution:

$$y' = (\text{derivative of first})(\text{second}) + (\text{first})(\text{derivative of second})$$

$$y' = 5x^4 \sin x + x^5 \cos x$$

Example 9

Differentiate $y = xe^x + \tan x + 7$.

Solution:

$$y' = (\text{derivative of first})(\text{second}) + (\text{first})(\text{derivative of second})$$
$$y' = (1)e^x + xe^x + \sec^2 x$$

Exercise 10

Find y' for $y = \frac{4}{\cos x} + \frac{5}{\tan x}$.

Solution:

$$y = 4\sec x + 5\cot x$$
$$y' = 4\sec x\tan x - 5\csc^2 x$$

Example 11

Differentiate $y = (\sin x + \cos x)\sec x$.

Solution:

$$
\begin{aligned}
y &= \sin x \sec x + \cos x \sec x \\
y &= \sin x \cdot \frac{1}{\cos x} + \cos x \cdot \frac{1}{\cos x} \\
y &= \tan x + 1 \\
{\color{red} y'} &= \sec^2 x
\end{aligned}
$$

Exercise 12

Find ${\color{red} y'}$ for $y = \tan x \cot x$.

Solution:

$$
\begin{aligned}
y &= \frac{\sin x}{\cos x} \cdot \frac{\cos x}{\sin x} \\
y &= 1 \\
{\color{red} y'} &= 0
\end{aligned}
$$

Example 13

Differentiate $y = \frac{\sec x}{1+\sec x}$.

Solution:

$$y' = \frac{(\text{denominator})(\text{derivative of numerator}) - (\text{numerator})(\text{derivative of de}}{(\text{denominator})^2}$$

$$y' = \frac{(1+\sec x)\sec x \tan x - \sec x \cdot \sec x \tan x}{(1+\sec x)^2}$$

$$y' = \frac{\sec x \tan x + \sec^2 x \tan x - \sec^2 x \tan x}{(1+\sec x)^2}$$

$$y' = \frac{\sec x \tan x}{(1+\sec x)^2}$$

Example 14

For which value(s) is the function defined by

$$f(x) = \begin{cases} ax + b, & x > \frac{\pi}{4} \\ \cos x, & x \leq \frac{\pi}{4} \end{cases}$$

differentiable at $x = \frac{\pi}{4}$?

Solution: We find

$$f'\left(\frac{\pi}{4}\right) = \lim_{h \to 0} \frac{f\left(\frac{\pi}{4} + h\right) - f\left(\frac{\pi}{4}\right)}{h} = \lim_{h \to 0} \frac{f(h)}{h}$$

We need to compute the left and right limit and we make them equal.

$$f\left(\frac{\pi}{4}^{-}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad\qquad f\left(\frac{\pi}{4}^{+}\right) = a$$

so we have $a = \frac{1}{\sqrt{2}}$.Now since the function is continuous, then we must have the right limit equal the left limit and so we have

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) \to \frac{\pi}{4\sqrt{2}} + b = \cos\frac{\pi}{4} \to b =$$

Exercise 15

For which value(s) is the function defined by

$$f(x) = \begin{cases} ax + b, & x > \frac{\pi}{6} \\ \tan x, & x \leq \frac{\pi}{6} \end{cases}$$

differentiable at $x = \frac{\pi}{6}$?