

Section 3.7

Implicit Differentiation

1 Lecture

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MATHS 101: Calculus I

Implicitly Defined Functions

So far, we have seen **explicitly defined functions**, i.e., functions of the form $y = f(x)$, where y is expressed totally in terms of x . Now we want to deal with implicitly defined functions such as the following:

Example 1

- 1 $x^2 + y^2 = 4.$
- 2 $e^{xy} - x^2 + 4y = 5x - 4.$
- 3 $\ln(y) + e^{2x} = y^2 e^{-x}.$

The above functions are said to be **implicitly defined** function. Note that sometimes, it is hard (or even impossible) to write y alone as a function of x .

Goal: To find the derivative y' of implicitly defined functions.

Example 2

Find y' for

$$y^2 = x$$

Solution 1:

We write y in term of x , so we get $|y| = \sqrt{x}$ and so $y = \pm\sqrt{x}$. Hence,

$$y' = \pm \frac{1}{2\sqrt{x}}$$

Solution 2: To find the derivative y' , we differentiate both sides with respect to x to get

$$2yy' = 1$$

$$y' = \frac{1}{2y}$$

Exercise 3

Find $\frac{dy}{dx}$ for $xy = 1$ by two ways, implicit and explicit.

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Example 4

Find y' for

$$x^2 + y^2 = 4$$

and use it to find an equation of the tangent line at $(1, \sqrt{3})$.

Solution: To find the derivative y' , we differentiate both sides with respect to x to get

$$2x + 2yy' = 0$$

$$2yy' = -2x \rightarrow y' = \frac{-2x}{2y}$$

$$y' = \frac{-x}{y}$$

$$m = y'|_{(1, \sqrt{3})} = \frac{-1}{\sqrt{3}}$$

Hence the equation of the tangent line is

$$y - y_1 = m(x - x_1) \rightarrow y - \sqrt{3} = \frac{-1}{\sqrt{3}}(x - 1) \rightarrow \sqrt{3}y + x = 4$$

Exercise 5

Find y' for

$$x + e^y = y + e^x$$

Solution: To find the derivative y' , we differentiate both sides with respect to x to get

$$1 + e^y \cdot y' = y' + e^x$$

$$e^y y' - y' = e^x - 1$$

$$(e^y - 1)y' = e^x - 1$$

$$y' = \frac{e^x - 1}{e^y - 1}$$

Example 6

Find y' for

$$xy^2 - 6x = 5 + 2y$$

Solution: To find the derivative y' , we differentiate both sides with respect to x to get

$$\begin{aligned}y^2 + 2xyy' - 6 &= 2y' \\2xyy' - 2y' &= -y^2 + 6 \\(2xy - 2)y' &= -y^2 + 6 \\y' &= \frac{-y^2 + 6}{2xy - 2}\end{aligned}$$

Exercise 7

Find y' for

$$xy = \cot(y)$$

Solution: To find the derivative y' , we differentiate both sides with respect to x to get

$$\begin{aligned}y + x \cdot y' &= -\csc^2(y) \cdot y' \\x \cdot y' + \csc^2(y) \cdot y' &= -y \\(x + \csc^2(y))y' &= -y \\y' &= \frac{-y}{(x + \csc^2(y))}\end{aligned}$$

Example 8

(The Folium of Descartes')

- (a) Find y' if $x^3 + y^3 = 6xy$.
- (b) Find an equation of the tangent line at $(3, 3)$.
- (c) Find an equation of the normal line at $(3, 3)$.

Solution: (a) To find the derivative y' , we differentiate both sides with respect to x to get

$$\begin{aligned}3x^2 + 3y^2 y' &= 6y + 6xy' \\3y^2 y' - 6xy' &= 6y - 3x^2 \\(3y^2 - 6x)y' &= 6y - 3x^2 \\y' &= \frac{6y - 3x^2}{3y^2 - 6x}\end{aligned}$$

Continue...

Example 9

(The Folium of Descartes')

(a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find an equation of the tangent line at $(3, 3)$.

Solution: (b) The equation of the tangent line is given by

$$y - y_1 = m(x - x_1)$$

We find the slope at $(3, 3)$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x}$$

$$m = \frac{6(3) - 3(3)^2}{3(3)^2 - 6(3)}$$

$$m = -1$$

Continue...

Hence we have

$$y - y_1 = m(x - x_1)$$

$$y - 3 = -(x - 3)$$

$$y + x = 6$$

Example 10

Find an equation of the tangent and normal lines at $(0, \pi)$ for the curve

$$x^2 \cos^2 y - \sin y = 0$$

Solution: To find the derivative y' , we differentiate both sides with respect to x to get

$$2x \cos^2 y + 2x^2 \cos y \sin y \cdot y' - \cos y \cdot y' = 0$$

$$2(0) \cos^2(\pi) + 2(0)^2 \cos(\pi) \sin(\pi) \cdot m - \cos(\pi) \cdot m = 0 \rightarrow m = 0$$

Hence the equation of the tangent line is

$$y - y_1 = m(x - x_1) \rightarrow y - \pi = 0(x - 1) \rightarrow \boxed{y = \pi} \text{ horizontal}$$

Hence the equation of the normal line is

$$\boxed{x = 0} \text{ vertical}$$

Example 11

Find $\frac{d^2y}{dx^2}$ for

$$xy + y - x = 4$$

Solution:

$$y + xy' + y' - 1 = 0 \rightarrow y' = \frac{1-y}{x+1}$$

$$y' + y' + xy'' + y'' = 0$$

$$(x+1)y'' = -2y'$$

$$y'' = \frac{-2y'}{x+1}$$

$$y'' = \frac{-2\frac{1-y}{x+1}}{x+1}$$

$$y'' = \frac{-2(1-y)}{(x+1)^2}$$

Exercise 12

Find y'' for $\sin y + \cos y = 1$