Section 3.8
Derivative of the inverse function and logarithms
3 Lecture

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MATHS 101: Calculus I

Topics

1. Inverse Functions (1 lecture).
2. Logarithms.
3. Derivative of the inverse function (1 lecture).
4. Logarithmic differentiation (1 lecture).

2- Logarithmic Function

Consider the exponential function $f(x) = a^x$.

Question: Does $f(x)$ has an inverse? Why?

Answer: Yes, by the horizontal line test.

- $f^{-1}(x)$ is called **logarithmic function** base a and it is denoted by

$$f^{-1}(x) = \log_a x$$

Note: (The fundamental equations)

1. $f(f^{-1})(x) = x$, so we have $a^{\log_a x} = x$.
2. $f^{-1}(f(x)) = x$, so we have $\log_a a^x = x$.

$$\underbrace{\log_a x = y}_{\text{logarithmic form}} \quad \text{if and only if} \quad \underbrace{x = a^y}_{\text{exponential form}}$$

If $a = e = 2.718281828\ldots$ (Euler number), then we simply write $\log_e$ as ln "ell en" and it is called the **natural logarithm**.

Properties of Logarithms

1. $\log_a(m \cdot n) = \log_a m + \log_a n$.
2. $\log_a(\frac{m}{n}) = \log_a m - \log_a n$.
3. $\log_a m^r = r \log_a m$.
4. $\log_a 1 = 0$.
5. $\log_a a = 1$.
6. (change of bases) $\log_a m = \frac{\log_b m}{\log_b a}$.

Exercise 1

Use the fundamental equations to prove these six properties of the logarithms.

Example 2

(Expansion) Write the following expression as sum or difference of logarithms

1. $\ln(\frac{x}{wz^2}) = \ln x - \ln(wz^2) = \ln x - (\ln w + \ln z^2) = \ln x - \ln w - 2\ln z.$
2. $\ln(\frac{x+1}{x+5})^4 = 4\ln(\frac{x+1}{x+5}) = 4(\ln(x+1) - \ln(x+5)).$
3. $\ln(\frac{\sqrt{x}}{(x^2)(x+3)^4}) = \ln\sqrt{x} - \ln x^2 - \ln(x+3)^4 =$
   $\ln x^{\frac{1}{2}} - 2\ln x - 4\ln(x+3) = \frac{1}{2}\ln x - 2\ln x - 4\ln(x+3) =$
   $-\frac{3}{2}\ln x - 4\ln(x+3).$

Exercise 3

Write each of the following expression as sum or difference of logarithms:
(1) $\log_3(\frac{5\cdot 7}{4})$ (2) $\log_2(\frac{x^5}{y^2})$ (3) $\log(\frac{x^2 z}{wy^2})$ (4) $\ln\sqrt{\frac{x+1}{x-2}}$.

Example 4

Write each of the following logarithm in terms of natural logarithm.

1. $\log_3 x = \frac{\ln x}{\ln 3}$.
2. $\log_6 7 = \frac{\ln 7}{\ln 6}$.
3. $\log_2 y = \frac{\ln y}{\ln 2}$.

The derivative of the inverse function

Strategy:

Goal: We want to find $\frac{d}{dx}\left(f^{-1}(x)\right)$.

$$\begin{aligned}
\text{Write } y = f^{-1}(x),&\ \text{we want to find } y' \\
f(y) &= f(f^{-1}(x)) \\
f(y) &= x \\
f'(y)\cdot y' &= 1 \\
y' &= \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}
\end{aligned}$$

Geometric Interpretation *

Note that

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$

so the slope of f^{-1} is reciprocal to the slope of f. Geometrically,

Example 5

Let $f(x) = x^3 - 3x^2 - 1$. Find $\frac{d}{dx}(f(x))$ and $\frac{d}{dx}\left(f^{-1}(x)\right)$ at the point $(3, -1)$

Solution:

$$\frac{d}{dx}(f(x)) = 3x^2 - 6x$$

$$\frac{d}{dx}(f(x))_{(3,-1)} = 3(3)^2 - 6(3) = 9$$

$$\begin{aligned}\frac{d}{dx}\left(f^{-1}(x)\right) &= \frac{1}{f'(y)}\\ &= \frac{1}{3y^2 - 6y}\end{aligned}$$

$$\frac{d}{dx}\left(f^{-1}(x)\right)_{(3,-1)} = \frac{1}{3(3)^2 - 6(3)} = \frac{1}{9}$$

Exercise 6

Let $f(x) = x + e^x$. What is the value of $f^{-1}(1)$. Find $(f^{-1})'(1)$.

Derivative of ln

Example 7

Find $\frac{d}{dx}(\ln x)$.

Solution:

$$y = \ln x$$
$$e^y = x$$
$$e^y \cdot y' = 1$$
$$y' = \frac{1}{e^y}$$
$$\boxed{y' = \frac{1}{x}}$$

Exercise 8

Find y' if $y = \log_a x$.
(Hint: Use the change of base formula to change it to ln)

Recall

The Chain Rule

Theorem 9

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

$$(f(g(x)))' = \textit{derivative of outer}(\textit{inner}) \cdot (\textit{derivative of inner})$$

Example 10

Find y' for each of the following:

1. $f(x) = \ln x^2 = \ln x^2 \to y' = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$
2. $f(x) = \ln(2x+3) = \ln(2x+3) \to y' = \frac{1}{(2x+3)} \cdot 2$
3. $f(x) = x \ln x \to y' = (1)\ln x + x \cdot \frac{1}{x} = \ln x + 1.$
4. $f(x) = \ln(\ln x) = \ln(\ln x) \to y' = \frac{1}{(\ln x)} \cdot \frac{1}{x}.$
5. $f(x) = \ln(\sin x) = \ln(\sin x) \to y' = \frac{1}{(\sin x)} \cdot \cos x = \cot x.$
6. $f(x) = \sin(\ln x) = \sin(\ln x) \to y' = \cos(\ln x)\frac{1}{(x)}.$

Exercise 11

Find the derivative of the following functions:

1. $y = \ln(\csc x - \cot x)$
2. $y = \frac{\ln x}{1+\ln x}$
3. $y = \ln\ln\ln x$

Derivative using the properties of Logarithms

Example 12

Find the derivative of

1. $f(x) = \ln x^{2017}$

Solution: First we re–write the function in terms using the properties of the ln to get a simplified function:

$$f(x) = 2017 \ln x$$

Hence

$$f'(x) = 2017 \frac{1}{x}$$

Exercise 13

Using the chain rule, find the derivative of the function of the previous example *without using the properties of the* ln, i.e., find $f'(x)$ for

$$f(x) = \ln(x^{2017})$$

Derivative using the properties of Logarithms

Example 14

Find the derivative of

1. $f(x) = \ln \sqrt[3]{\frac{x^3-1}{x^3+1}}$

Solution: First we re–write the function in terms using the properties of the ln to get a simplified function:

$$
\begin{aligned}
f(x) &= \ln \left(\frac{x^3-1}{x^3+1}\right)^{\frac{1}{3}} \\
&= \frac{1}{3}\left(\ln(x^3-1) - \ln(x^3+1)\right)
\end{aligned}
$$

Continue...

We write the inner function in blue and the outer function in red and we apply the chain rule.

$$\text{derivative of outer}\,(\text{inner})\cdot(\text{derivative of inner})$$

$$f(x) = \frac{1}{3}\left(\ln(x^3 - 1) - \ln(x^3 + 1)\right)$$

$$f'(x) = \frac{1}{3}\left(\frac{1}{x^3 - 1}\cdot(3x^2) - \frac{1}{x^3 + 1}\cdot(3x^2)\right)$$

Exercise 15

Using the chain rule, find the derivative of the function of the previous example *without using the properties of the* ln, i.e., find $f'(x)$ for

$$f(x) = \ln\left(\sqrt[3]{\frac{x^3-1}{x^3+1}}\right)$$

Example 16

Find $\frac{d^4y}{dx^4}$ for

$$y = 5\ln x$$

Solution:

$$\begin{aligned} y' &= 5\frac{1}{x} \\ &= 5x^{-1} \\ y'' &= -5x^{-2} \\ y''' &= 10x^{-3} \\ y^{(4)} &= -30x^{-4} \\ &= \frac{-30}{x^4} \end{aligned}$$