Section 4.1
Relative Extrema
3 Lectures

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MATHS 101: Calculus I

Application of Differentiation

One of the most important applications of differential calculus are the *optimization problems*, i.e., finding the optimal (best) way to do something. In our case, these optimization problem are reduced to find the minimum or maximum of a function.

Example 1

1. Find the length that maximizes the area.
2. Find the radius that minimize the perimeter of certain circle.

1 - Monotone Functions

Increasing Function

Geometry

Algebra

If $a \leq b$, then $f(a) \leq f(b)$

Exercise 2

Write a similar definition for *decreasing* function.

Definition 3

A *monotone* function is either an increasing or decreasing function.

Question: How to tell when a function is increasing or decreasing?
Answer: One way is to use the definition above, which is hard to do in general. The other way is to use Calculus as follows:

- If $f'(x) \geq 0$, then $f(x)$ is increasing.
- If $f'(x) \leq 0$, then $f(x)$ is decreasing.

2 - Absolute Extrema

Absolute Maximum (Global Maximum)

Geometry

Algebra

$f(c)$ is an *absolute maximum* (*global maximum*) if

$$f(x) \leq f(c), \text{ for all } x$$

- $f(c)$ is the absolute maximum (only one).
- c is called absolute maximizer

Exercise 4

Write a similar definition for *absolute minimum*.

Definition 5

An *absolute extrema* is either an absolute maximum or absolute minimum function.

3 - Relative Extrema

Relative Maximum (Local Maximum)

Geometry

Algebra

$f(c)$ is an *local maximum* (*relative maximum*) if

$$f(x) \leq f(c), \text{ for some value of } x \text{ nea}$$

- $f(c)$ is the local maximum (maybe more than one).
- c is called local maximizer

Exercise 6

Write a similar definition for *local minimum*.

Definition 7

An *local extrema* (*relative extrema* is either an local maximum or loca minimum function.

Example 8

Let $f(x) = \sin x$, then

- It has a global minimum at $(\frac{-\pi}{2}, -1), (\frac{3\pi}{2}, -1), \ldots, (\frac{3\pi}{2} + 2n\pi, -1)$.
- It has a global maximum at $(\frac{-3\pi}{2}, 1), (\frac{\pi}{2}, 1), \ldots, (\frac{\pi}{2} + 2n\pi, 1)$.

Example 9

Let $f(x) = x^2$, then

- It has a global minimum at (0,0).
- It has no global maximum.

Example 10

Let $f(x) = e^x$, then

- It has no global minimum.
- It has no global maximum.

critical numbers

Question: How to find the extrema (local min, local max, absolute min, absolute max)?

Answer: The following are the candidates for the extrma.

Definition 11

A number c is called a critical number for $f(x)$ if either

$$f'(c) = 0 \text{ or } f'(c) \text{ does not exist}$$

Note: These critical numbers are the candidates for local maximum or local minimum.

To find these numbers, write the derivative as rational function, i.e., $f'(x) = \frac{\text{numerator}}{\text{denominator}}$ and then we have

1. $f'(x) = 0 \rightarrow \text{numerator} = 0.$
2. $f'(x) = \text{ does not exist} \rightarrow \text{denominator} = 0.$

Example 12

Find the critical numbers of the following function

$$f(x) = x^3 + x^2 - x$$

Solution:
We find the derivative first which is

$$f'(x) = 3x^2 + 2x - 1$$

To find the critical numbers, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$
$$\text{numerator} = 0$$
$$3x^2 + 2x - 1 = 0$$
$$x = -1 \text{ or } x = \frac{1}{3}$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Exercise 13

Find the critical numbers of the following function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

Example 14

Find the critical numbers of the following function

$$f(x) = \sqrt{3}\sin x + \cos x$$

Solution:
We find the derivative first which is

$$f'(x) = \sqrt{3}\cos x - \sin x$$

We find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$
$$\text{numerator} = 0$$
$$\sqrt{3}\cos x - \sin x = 0$$
$$\sqrt{3}\cos x = \sin x$$
$$\sqrt{3} = \tan x \to x = \tan^{-1}\sqrt{3}$$
$$x = \frac{\pi}{3} + 2n\pi \text{ or } x = \frac{4\pi}{3} + 2n\pi$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Exercise 15

Find the critical numbers of the following function

$$f(x) = \frac{x}{2} - \sin x$$

Example 16

Find the critical numbers of the following function

$$f(x) = \sqrt{1 - x^2}$$

Solution:
We find the derivative first which is

$$f'(x) = \frac{-2x}{2\sqrt{1 - x^2}}$$

To find the critical numbers, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$
$$\text{numerator} = 0$$
$$-2x = 0$$
$$x = 0$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 - x^2 = 0$$
$$x = 1 \text{ or } x = -1$$

Exercise 17

Find the critical numbers of the following function

$$f(x) = \sqrt[3]{4 - x^2}$$

Example 18

Find the critical numbers of the following function

$$f(x) = \frac{x-1}{x^2-x+1}$$

Solution:

We find the derivative first which is

$$f'(x) = \frac{(x^2-x+1)(1)-(x-1)(2x-1)}{(x^2-x+1)^2} = \frac{-x^2+2x}{(x^2-x+1)^2}$$

To find the critical numbers, we find where the derivative equal to zero or does not exist.

$f'(x) = 0$

numerator $= 0$

$-x^2+2x = 0$

$x = 0$ or $x = 2$

$f'(x)$ does not exist

denominator $= 0$

$x^2-x+1 = 0$

No Solution

Exercise 19

Find the critical numbers of the following function

$$f(x) = \frac{x-1}{x^2-4}$$