Section 4.6
Optimization Problem
2 Lectures

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MATHS 101: Calculus I

Second Derivative Test

Theorem 1

(Second Derivative Test) Suppose $f'(c) = 0$, *then*

1. *If* $f''(c) < 0$, *then* f *has a local maximum at* c.
2. *If* $f''(c) > 0$, *then* f *has a local minimum at* c.

Notes:

- If $f''(c) = 0$, then we say the second derivative test is inconclusive! and in that case we need to use the first derivative test.
- This is very useful for this section to check quickly if we have a maximizer or minimizer.

Example 2

Find the local maximum and local minimum (if any) using the second derivative test.

$$f(x) = x^3 - 12x + 1$$

Solution:

We find the derivatives first which are

$$f'(x) = 3x^2 - 12$$
$$f''(x) = 6x$$

To find the critical points, we find where the first derivative equal to zero or does not exist.

$$f'(x) = 0$$
$$\text{numerator} = 0$$
$$3x^2 - 12 = 0$$
$$x = -2 \text{ or } x = 2$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Second Derivative Test

$$f''(-2) = 6(-2) = -12 < 0$$

So $x = -2$ is a local maximizer with maximum $f(-2) =$.

$$f''(2) = 6(2) = -2 > 0$$

So $x = 2$ is a local minimizer with minimum $f(2) =$.

Exercise 3

Find the local maximum and local minimum (if any) using the second derivative test.

$$f(x) = 7 - 2x^4$$

Solution:

We find the derivatives first which are

$$f'(x) = -8x^3$$
$$f''(x) = -24x^2$$

To find the critical points, we find where the first derivative equal to zero or does not exist.

$$f'(x) = 0$$
$$\text{numerator} = 0$$
$$-8x^3 = 0$$
$$x = 0$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Second Derivative Test

$$f''(0) = -246(0) = 0$$

So the second derivative test is inconclusive! and we apply the first derivative test.

Example 4

Let $f(x)$ be continuous function and have critical numbers 1 and 2. The second derivative is given by $f''(x) = 3x^2 - 16x + 17$. Where does f have a local maximum or local minimum?

Solution:

$$f''(1) = 3(1)^2 - 16(1) + 17 = 4 > 0$$

So $x = 1$ is a local minimizer.

$$f''(2) = 3(2)^2 - 16(2) + 17 = -3$$

So $x = 2$ is a local maximizer.

Example 5

What is the smallest perimeter possible for a rectangle whose area is 4 cm^2? What are the dimensions?

Solution:

Given: Area $= 4 \Rightarrow A = 4 = \ell w \Rightarrow \ell = \frac{4}{w}$

Required: Minimize Perimeter

$$\text{Required: Minimize } P = 2\ell + 2w$$
$$P = 2\frac{4}{w} + 2w$$
$$P = \frac{8}{w} + 2w$$

We find the derivatives first which are

$$P' = \frac{-8}{w^2} + 2 = \frac{-8 + 2w^2}{w^2}$$
$$P'' = \frac{16}{w^3}$$

Critical Points

To find the critical points, we find where the first derivative equal to zero or does not exist.

$$\begin{aligned} P'(x) &= 0 \\ \text{numerator} &= 0 \\ -8 + 2w^2 &= 0 \\ w = 2 \text{ or } w &= -2\,(\text{rejected}) \end{aligned}$$

$$\begin{aligned} f'(x) & \text{ does not exist} \\ \text{denominator} &= 0 \\ w^2 &= 0 \\ w &= 0\,(\text{rejected}) \end{aligned}$$

Maximum or Minimum

To find out whether $w = 2$ is a maximizer or minimizer, we will use the second derivative test. so we check

$$P''(2) = \frac{16}{(2)^3} > 0$$

So $w = 2$ cm is a local minimizer with minimizer length $\ell = \frac{4}{2} = 2$ cm and minimum perimeter $P(2) = 8$ cm.

Exercise 6

Find the largest area of a rectangle whose perimeter is 32 cm.

Example 7

Determine the dimensions of the rectangle of the largest area that can be inscribed in the right triangle with sides 3,4, and 5?.

Solution:

Given: $\tan\theta = \frac{4}{3} = \frac{\ell}{3-w} \Rightarrow 4(3-w) = 3\ell \Rightarrow \ell = \frac{4}{3}(3-w)$

$$\text{Required: Maximize Area}$$
$$\text{Required: Maximize } A = \ell w$$
$$P = \frac{4}{3}(3-w)w$$
$$P = 4w - \frac{4}{3}w^2$$

We find the derivatives first which are

$$A' = 4 - \frac{8}{3}w$$
$$P'' = \frac{-8}{3}$$

Critical Points

To find the critical points, we find where the first derivative equal to zero or does not exist.

$$\begin{aligned} A'(x) &= 0 \\ \text{numerator} &= 0 \\ 4 - \frac{8}{3}w &= 0 \\ w &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} A'(x) &\text{ does not exist} \\ \text{denominator} &= 0 \\ 1 &= 0 \end{aligned}$$

Always False

No Solution

Maximum or Minimum

To find out whether $w = \frac{3}{2}$ is a maximizer or minimizer, we will use the second derivative test. so we check

$$A''\left(\frac{3}{2}\right) = \frac{-8}{3} < 0$$

So $w = \frac{3}{2}$ cm is a local maximizer with maximizer length $\ell = \frac{4}{3}(3 - \frac{3}{2}) = 2$ cm and maximum perimeter $P(2) = 3$ cm^2.

Example 8

Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$

Solution:
Given: $y = \sqrt{x}$

Required: minimum distance between the curve and the point

Required: minimize $D = (x-3)^2 + (y-0)^2$

$$D = x^2 - 6x + 9 + (\sqrt{x})^2$$
$$D = x^2 - 5x + 9$$

We find the derivatives first which are

$$D' = 2x - 5$$
$$D'' = 2$$

Critical Points

To find the critical points, we find where the first derivative equal to zero or does not exist.

$$
\begin{aligned}
D'(x) &= 0 \\
\text{numerator} &= 0 \\
2x - 5 &= 0 \\
x &= \frac{5}{2}
\end{aligned}
$$

$D'(x)$ does not exist

$$
\begin{aligned}
\text{denominator} &= 0 \\
1 &= 0
\end{aligned}
$$

Always False

No Solution

Maximum or Minimum

To find out whether $x = \frac{5}{2}$ is a maximizer or minimizer, we will use the second derivative test. so we check

$$D''\left(\frac{5}{2}\right) = 2 > 0$$

So $x = \frac{5}{2}$ is a local minimizer with a minimum distance $D = 2.75$.

Exercise 9

Find the point on the curve $y^2 = \frac{1}{2}x^3$ that is closest to the point $(5, 0)$

Example 10

What is the minimum vertical distance between the curves $y = x^2 + 1$ and $y = x - x^2$.

Solution:

$$\text{Required: minimum distance}$$

$$\text{Required: minimize } D = (x^2 + 1) - (x - x^2)$$

$$D = 2x^2 - x + 1$$

We find the derivatives first which are

$$D' = 4x - 1$$

$$D'' = 4$$

Critical Points

To find the critical points, we find where the first derivative equal to zero or does not exist.

$$
\begin{aligned}
D'(x) &= 0 \\
\text{numerator} &= 0 \\
4x - 1 &= 0 \\
x &= \frac{1}{4}
\end{aligned}
$$

$D'(x)$ does not exist

$$
\begin{aligned}
\text{denominator} &= 0 \\
1 &= 0
\end{aligned}
$$

Always False

No Solution

Maximum or Minimum

To find out whether $x = \frac{1}{4}$ is a maximizer or minimizer, we will use the second derivative test. so we check

$$D''\left(\frac{1}{4}\right) = 4 > 0$$

So $x = \frac{1}{4}$ is a local minimizer with a minimum distance $D = \frac{13}{16}$.

Exercise 11

What is the minimum vertical distance between the curves $y = 3x^2 + 6x + 8$ and $y = 2x + 2$.

Example 12

Find a number for which the sum of it and its reciprocal is the smallest possible.

Solution:

Given: Nothing

Required: Minimize Sum

$$\text{Required: Minimize } S = x + \frac{1}{x}$$

We find the derivatives first which are

$$S' = 1 + \frac{-1}{x^2} = \frac{x^2 - 1}{x^2}$$

$$P'' = \frac{2}{x^3}$$

Critical Points

To find the critical points, we find where the first derivative equal to zero or does not exist.

$$\begin{aligned} S'(x) &= 0 \\ \text{numerator} &= 0 \\ x^2 - 1 &= 0 \\ x = 1 &\text{ or } x = -1 \end{aligned}$$

$$\begin{aligned} S'(x) &\text{ does not exist} \\ \text{denominator} &= 0 \\ x^2 &= 0 \\ x &= 0\,(\text{rejected}) \end{aligned}$$

Maximum or Minimum

To find out whether $x = 1$ or $x = -1$ is a maximizer or minimizer, we will use the second derivative test. so we check

$$S''(1) = \frac{2}{(1)^3} > 0$$

So $x = 1$ cm is a local minimizer with minimum sum $S(1) = 2$ cm.

Exercise 13

Find a positive number for which the sum of its reciprocal and four times its square is the smallest possible.

Example 14

For what values of a make the function

$$f(x) = x^2 + ax$$

have a local minimum at $x = 2$.

Solution:
We find the derivatives first which are

$$f'(x) = 2x + a$$

$$f \text{ has a local minimum at } x = 2 \Rightarrow f'(2) = 0 \Rightarrow 4 + a = 0$$

Solving the above equation yield that $a = -4$

Example 15

For what values of a and b make the function

$$f(x) = x^3 + ax^2 + bx$$

have a local maximum at $x = -1$ and local minimum at $x = 3$.

Solution:
We find the derivatives first which are

$$f'(x) = 3x^2 + 2ax + b$$

$$f \text{ has a local mamximum at } x = -1 \Rightarrow f'(-1) = 0 \Rightarrow 3 - 2a + b = 0$$

$$f \text{ has a local minimum at } x = 3 \Rightarrow f'(3) = 0 \Rightarrow 27 + 6a + b = 0$$

Solving the above two equations yield that $a = 3$ and $b = 3$

Exercise 16

Solve the previous example, but this time, assume the function has a local minimum at $x = 4$ and a point of inflection at $x = 1$.