Section 4.8
Anti Derivative and Indefinite Integrals
2 Lectures

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MATHS 101: Calculus I

Indefinite Integral

Given a function f, if F is a function such that

$$F'(x) = f(x)$$

then F is called antiderivative of f.

Definition 1

An antiderivative of f is simply a function whose derivative is f.

Note: Any two antiderivatives of a function differ only by a constant.

Indefinite Integrals

If $F(x)$ is the antiderivative of $f(x)$, we will write

$$\int f(x)\,dx = \underbrace{F(x)}_{\text{antiderivative}} + C$$

where

- The symbol $\int$ is called the **integral sign**.
- The function $f(x)$ is called the **integrand**.
- The constant C is called the **constant of integration**.
- dx indicates the variable involved in the integration which is x.

Note: The Fundamental Theorem of Calculus

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x) \text{ and } \int \frac{d}{dx}\left(f(x)\right)\,dx = f(x)$$

Integration and differentiation are reversing each other.

Examples

Example 2

Find $\int 7 \, dx$.

Solution: We need to find what is the function that if we differentiate it we get 7?

$$= 7x + C$$

Example 3

Find $\int x \, dx$.

Solution: We need to find what is the function that if we differentiate it we get x?

$$= \frac{1}{2}x^2 + C$$

Examples

Example 4

Find $\int x^9 \, dx$.

Solution: We need to find what is the function that if we differentiate it we get x^9?

$$= \frac{1}{10}x^{10} + C$$

Example 5

Find $\int \frac{1}{x} \, dx$.

Solution: We need to find what is the function that if we differentiate it we get $\frac{1}{x}$?

$$= \ln x + C$$

Elementary integration formula

1. $\int k\,dx = kx + C$.

2. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \qquad n \neq -1$.

3. $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C \qquad x > 0$.

4. $\int e^x\,dx = e^x + C$.

5. $\int kf(x)\,dx = k\int f(x)\,dx$.

6. $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$.

Example 6

Find $\int 5x^{-9}\, dx$.

Solution:

$$= \frac{5}{-8}x^{-8} + C$$

Exercise 7

Find $\int \frac{3}{x^6}\, dx$.

Solution:

$$\int \frac{3}{x^6}\, dx = \int 3x^{-6}\, dx = \frac{3}{-5}x^{-5} + C$$

Example 8

Find $\int 4x^6 + 3x^4 + 2x + 9 + \frac{1}{x}\,dx$.

Solution:

$$= \frac{4}{7}x^7 + \frac{3}{5}x^5 + x^2 + 9x + \ln|x| + C$$

Exercise 9

Find $\int x^{9.9} - 7x^6 + 3x^{-4} + x^{-1} + \sqrt{2}\,dx$.

Solution:

$$= \frac{1}{10.9}x^{10.9} - x^7 + \frac{3}{-3}x^{-3} + \ln|x| + \sqrt{2}x + C$$

Example 10

Find $\int \sqrt{x} + \frac{5}{3\sqrt[3]{x^2}}\, dx$.

Solution:

$$= \int x^{\frac{1}{2}} + \frac{5}{3}x^{\frac{-2}{3}}\, dx = \frac{2}{3}x^{\frac{3}{2}} + \frac{5}{3}3x^{\frac{1}{3}} + C$$

Exercise 11

Find $\int e^x + x^e + e^2\, dx$.

Solution:

$$= e^x + \frac{1}{e+1}x^{e+1} + e^2x + C$$

Example 12

Find $\int x^{-2}(4x^3 + 3x + 5)\,dx$.

Solution:

$$= \int 4x + 3x^{-1} + 5x^{-2}\,dx = 2x + 3\ln x - 5x^{-1} + C$$

Exercise 13

Find $\int \frac{x^4+10x}{x^2}\,dx$.

Solution:

$$= \int x^2 + 10x^{-1}\,dx = \frac{1}{3}x^3 + 10\ln|x| + C$$

Example 14

Find $\int (x+2)^2 \, dx$.

Solution:

$$= \int x^2 + 4x + 4 \, dx = \frac{1}{3}x^3 + 2x^2 + 4x + C$$

Exercise 15

Find $\int \frac{d}{dx}\left(\frac{1}{\sqrt{1+x^3}}\right) \, dx$.

Solution:

$$\int \frac{d}{dx}\left(\frac{1}{\sqrt{1+x^3}}\right) \, dx = \frac{1}{\sqrt{1+x^3}} + C$$

Exercise 16

Find $\int (7x^3 - 6x^2 - \ln 3)\, dx$.

Solution:

$$= \frac{7}{4}x^4 - 2x^3 - (\ln 3)x + C$$

Exercise 17

Find $\int e^{\ln(x^2+1)}\, dx$.

$$\int e^{\ln(x^2+1)}\, dx = \int (x^2+1)\, dx = \frac{x^3}{3} + x + C$$

Exercise 18

Find $\int \, dx$.

$$\int dx = \int 1\, dx = x + C$$

Indefinite Integrals involving Trigonometric and inverse trigonometric functions

Example 19

Find $\int \csc^2 x \, dx$.

Solution:

$$= -\cot x + C$$

Exercise 20

Find $\int 4 \sin x + 3 \cos x \, dx$.

Solution:

$$= -4 \cos x + 3 \sin x + C$$

Example 21

Find $\int \frac{x^3 - x^4\cos x + 5}{x^4}\,dx$.

Solution:

$$= \int \left(\frac{1}{x} - \cos x + 5x^{-4}\right) dx$$

$$= \ln|x| - \sin x - \frac{5}{2}x^{-3} + C$$

Exercise 22

Find $\int \frac{7}{\sqrt{1-x^2}}\,dx$.

Solution:

$$= 7\sin^{-1} x + C$$

Example 23

Find $\int \frac{1+\cos^2 x}{\cos^2 x}\, dx$.

Solution:

$$\begin{aligned}\int \frac{1+\cos^2 x}{\cos^2 x} &= \int\left(\frac{1}{\cos^2 x}+1\right)\, dx\\ &= \int \sec^2 x + 1\, dx\\ &= \tan x + x + C\end{aligned}$$

Exercise 24

Find $\int \frac{\sin 2x}{\sin x}\,dx$.

Solution:

$$\begin{aligned}\int \frac{\sin 2x}{\sin x}\,dx &= \int \frac{2\sin x\cos x}{\sin x}\,dx\\ &= \int 2\cos x\,dx\\ &= 2\sin x + C\end{aligned}$$

Example 25

Find $\int \frac{t^2-1}{t^4-1}\,dx$.

Solution:

$$\begin{aligned}\int \frac{t^2-1}{t^4-1} &= \int \frac{(t^2-1)}{(t^2+1)(t^2-1)}\,dx \\ &= \int \frac{1}{t^2+1}\,dx \\ &= \tan^{-1} t + C\end{aligned}$$

Exercise 26

Find $\int \cos x(\tan x + \sec x)\, dx$.

Solution:

$$\begin{aligned}
\int \cos x(\tan x + \sec x)\, dx &= \int \cos x \tan x + \cos x \sec x \, dx \\
&= \int \cos x \frac{\sin x}{\cos x} + \cos x \frac{1}{\cos x}\, dx \\
&= \int (\sin x + 1)\, dx \\
&= -\cos x + x + C
\end{aligned}$$

Note: If $\int f(x)\,dx = F(x) + C$, then $\int f(kx)\,dx = \frac{1}{k}F(kx) + C$.

Example 27

1. $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C.$
2. $\int e^{-x}\,dx = \frac{1}{-1}e^{-x} + C.$
3. $\int e^{-2x}\,dx = \frac{1}{-2}e^{-2x} + C.$

Example 28

Find $\int (e^x + 5)^2 \, dx$.

Solution:

$$\int (e^x + 5)^2 \, dx = \int (e^{2x} + 10e^x + 25) \, dx$$
$$= \frac{1}{2}e^{2x} + 10e^x + 25x + C$$

Exercise 29

Find $\int (e^x - e^{-x})^2 \, dx$.

Example 30

Find $\int \frac{1+e^x}{e^x}\,dx$.

Solution:

$$\begin{aligned}\int \frac{1+e^x}{e^x}\,dx &= \int\left(\frac{1}{e^x}+\frac{e^x}{e^x}\right)\,dx\\ &= \int (e^{-x}+1)\,dx\\ &= -e^{-x}+x+C\end{aligned}$$

Example 31

1. $\int \sin(2x)\,dx = -\frac{1}{2}\cos(2x) + C.$
2. $\int \cos(-x)\,dx = \frac{1}{-1}\sin(-x) + C.$
3. $\int \sec(\pi x)\tan(\pi x)\,dx = \frac{1}{\pi}\sec(\pi x) + C.$
4. $\int \sec^2(7x)\,dx = \frac{1}{7}\tan(7x) + C.$

Recall: The trigonometric Identitites:

(cos and sin functions)

1. $\cos^2 x + \sin^2 x = 1.$
2. $1 - \sin^2 x = \cos^2 x.$
3. $1 - \cos^2 x = \sin^2 x.$

(tan and sec functions)

1. $\sec^2 x - \tan^2 x = 1.$
2. $\sec^2 = 1 + \tan^2 x.$
3. $\tan^2 x = \sec^2 x - 1.$

(Double Angle Formula)

1. $\sin(2x) = 2\sin x \cos x.$
2. $\cos(2x) = \cos^2 x - \sin^2 x.$
3. $\cos^2 x = \frac{1}{2}(1 + \cos(2x)).$
4. $\sin^2 x = \frac{1}{2}(1 - \cos(2x)).$

Example 32

Find $\int \sec^2 x \, dx$.

Solution:

$$\int \sec^2 x \, dx = \tan x + C$$

Example 33

Find $\int \tan^2 x \, dx$.

Solution:

$$\begin{aligned} \int \tan^2 x \, dx &= \int (\sec^2 x - 1) \, dx \\ &= \tan x - x + C \end{aligned}$$

Example 34

Find $\int \sin^2 x \, dx$.

Solution:

$$
\begin{aligned}
\int \sin^2 x \, dx &= \int \frac{1}{2}\left(1-\cos(2x)\right)\, dx \\
&= \frac{1}{2}\int \left(1-\cos(2x)\right)\, dx \\
&= \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + C
\end{aligned}
$$

Exercise 35

Find $\int \cos^2 x \, dx$

Exercise 36

Find $\int 6 - \cot^2 x \, dx$.

Solution:

$$\begin{aligned}\int 6 - \cot^2 x \, dx &= \int 6 - (\csc^2 x - 1) \, dx \\ &= \int 7 - \csc^2 x \, dx \\ &= 7x - \cot x + C\end{aligned}$$

Example 37

Find $\int (\sec x + \tan x)^2 \, dx$.

Solution:

$$\begin{aligned}
\int (\sec x + \tan x)^2 \, dx &= \int \sec^2 x + 2\sec x \tan x + \tan^2 x \, dx \\
&= \int \sec^2 x + 2\sec x \tan x + \sec^2 x - 1 \, dx \\
&= \int 2\sec^2 x + 2\sec x \tan x - 1 \, dx \\
&= 2\tan x + 2\sec x - x + C
\end{aligned}$$

Example 38

Find $\int \frac{1}{\sqrt{4-x^2}}\, dx$.

Solution:

$$\int \frac{1}{\sqrt{4-x^2}}\, dx = \sin^{-1}\left(\frac{x}{2}\right)$$

Exercise 39

Find $\int \frac{1}{9+x^2}\, dx$.

Solution:

$$\int \frac{1}{9+x^2}\, dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$$