Extra
2 Lectures

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MATHS 101: Calculus I

3 - Intermediate Value Theorem (Application of Calculus)

Here we give an application of calculus to finding the location of a root to an equation.

Definition 1

A number c is called a **root** (**zero**) for a function f if

$$f(c) = 0$$

Theorem 2

Let f be a continuous function on an interval $[a, b]$ such that $f(a)$ and $f(b)$ have different signs , then there exist a root $c \in (a, b)$ such that $f(c) = 0$.

Example 3

Show there exists a root for $x^3 - x - 1 = 0$ between 1 and 2.

Solution: Note that the function is continuous (polynomial) and we have

$$f(1) = -1 < 0 \quad f(2) = 5 > 0$$

Therefore, by the IVT, there must be a root $c \in (1, 2)$ such that $f(c) = 0$. The IVT does not tell us how to find that root.

Exercise 4

Show there exists a root for $x^3 - 3x - 1 = 0$.

Solution: Note that the function is continuous (polynomial). Here we do not have an interval, so we need to find a suitable interval (two end–points with different sign). One choice is 0, so we have $f(0) = -1 < 0$. Now we look for some other number with a positive value, for example, $f(2) = 1 > 0$. Therefore, by the IVT, there must be a root $c \in (0, 2)$ such that $f(c) = 0$. The IVT does not tell us how to find that root.

Example 5

Show there exist a number $c \in (0,1)$ such that $\sqrt[3]{c} = 1 - c$.

Solution: The problem can be translate as to find a number $c \in (0,1)$ such that $\sqrt[3]{c} - 1 + c = 0$, i.e., we need to show that there is a root for the equation $\sqrt[3]{x} - 1 + x = 0$. Note that the function is continuous. We have $f(0) = -1 < 0$ and $f(1) = 1 > 0$ Therefore, by the IVT, there must be a root $c \in (0,1)$ such that $f(c) = 0$, i.e., we have $\sqrt[3]{c} = 1 - c$.

Exercise 6

(Challenging Problem) **The fixed point theorem** Suppose f is a continuous function on $[0,1]$ such that $0 \leq f(x) \leq 1$. Show there exist $c \in (0,1)$ such that $f(c) = c$.

(Hint: Apply IVT to $g(x) = f(x) - x$).

Netwon's Method

Recall:

Theorem 7

(Intermediate Value Theorem) Let f be a continuous function on an interval $[a, b]$ such that $f(a)$ and $f(b)$ have different signs, then f must have a root in $[a, b]$, i.e., there exists $c \in [a, b]$ such that $f(c) = 0$.

Recall that the intermediate value theorem tells us that there will be a root, but does not tell us how to find it. We will use Newton's method to approximate the root.

1. Guess an initial approximation x_0 to the root.
2. Determine a new approximation using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example 8

Consider the equation $f(x) = x^7 + 2x^5 + x^3 + 14x + 1$.

1. Use the intermediate value theorem to show that the equation above has **a** root.
2. Use Rolle's theorem theorem to show that the equation above has exactly **one** root.
3. Use the Newton's method to approximate a root of the equation above with $x_0 = 1$.

Example 9

Consider the equation $x^2 - 2 = 0$.

1. Use the intermediate value theorem to show that the equation above has a root in $[1, 2]$.
2. Use the Newton's method to approximate a root of the equation above in $[1, 2]$ with $x_0 = 1$.

Solution: (1) Let $f(x) = x^2 - 2$ (continuous) and we have $f(1) = -1 < 0$ while $f(2) = 2 > 0$, therefore by the intermediate value theorem, there must be a root in $[1, 2]$.
(2) Let $x_0 = 1$, we have $f'(x) = 2x$ and

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(1)^2 - 2}{2(1)} = 1.5$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{(1.5)^2 - 2}{2(1.5)} =$$

$$x_3 = x_3 - \frac{f(x_2)}{f'(x_2)} = - \frac{()^2 - 2}{2()} =$$