Preliminaries
$2\frac{1}{2}$ Lectures

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MATHS 101: Calculus I

Pre Calculus $\rightarrow$ MATHS 101: Calculus

- MATHS 101 is all about functions!
- MATHS 101 is an introductory course to a branch of Mathematics called Calculus.

Calculus

Differentiation

- We want to find the derivative of a function, which is finding the slope of the tangent line to the graph of a function at a given point.

Integration

- We want to find the integrate a function, which is finding the area under the graph of a function on a given interval.

Questions

Question 1 What is the relation between differentiation and integration? In other words, what is the relation between finding the slope of the tangent line and finding the area under the curve of a function?

Question 2 Why they are given together at the same course while they might look as two different branches of mathematics? (one measures the slope and the other measures the area)?

Answer The connection is given in the fundamental theorem of calculus which states (informally) that differentiation and integration are reversing each other! (In fact, both can be defined in terms of a limit!)

In MATHS 101, we will study

1. Limit of a function.
2. Derivative and its applications.
3. Integration and its applications.

Note: We want to differentiate (integrate) all kind of functions. So in MATHS 101, the strategy will be

1. Find the derivative (integral) of the basic functions, e.g., x^n,c, e^x, a^x, $\ln x$, $\log_a x$, $\sin x$, $\cos x$, $\tan x$, $\sin^{-1} x$, etc.
2. Establish rules to find the derivative (integral) of the new functions from the basic ones, i.e., rules for the sum, difference, product, quotient, composite, inverse, etc.

Hope you will have a nice course

Preliminaries: (From High school)

In this lecture, we will go over some important topics from high school. These are

1. Functions.
2. Graphs.
3. Lines.
4. Factoring.

1. Definition of a function

A **function** from a set X to a set Y is an *assignment* (*rule*) that tells how one element x in X is related to only one element y in Y.

Notation:

- $f : X \to Y$.
- $y = f(x)$.   "f of x".
- x is called the input (independent variable) and y is called the output (dependent variable).
- The set X is called the domain and Y is called the co–domain. While the set of all outputs is called the range.

Think about the function as a vending machine!

Question: How to describe a function mathematically?

Answer: By using algebraic formula!

Example 1

Consider the function

$$\begin{aligned} f : (-\infty, \infty) &\to (-\infty, \infty) \\ x &\mapsto 3x + 1 \end{aligned}$$

or simply by $f(x) = 3x + 1$

- f(1)=3(1)+1=4.
- f(0)=3(0)+1=1.
- f(-2)=3(-2)+1=-5.
- f(-7)=3(-7)+1=-20.

- Domain $= (-\infty, \infty)$.
- Co–domain$=(-\infty, \infty)$.
- Range$=(-\infty, \infty)$.

Exercise 2

$$f : (-\infty, \infty) \to (-\infty, \infty)$$
$$x \mapsto x^2$$

or simply by $f(x) = x^2$

- f(1)=$(1)^2$=1.
- f(0)=$(0)^2$=0.
- f(-1)=$(-1)^2$=1.
- f(-2)=$(-2)^2$=4.
- f(-4)=$(-4)^2$=16.
- f(4)=$(4)^2$=16.

- Domain $= (-\infty, \infty)$.
- Co–domain=$(-\infty, \infty)$.
- Range=$[0, \infty)$.

Example 3

$$f : (-\infty, \infty) \to (-\infty, \infty)$$
$$x \mapsto \frac{1}{x}$$

or simply by $f(x) = \frac{1}{x}$

- f(1)=$\frac{1}{1}$=1.
- f(-1)=$\frac{1}{-1}$=-1.
- f(2)=$\frac{1}{2}$=$\frac{1}{2}$.
- f(-4)=$\frac{1}{-4}$=$\frac{-1}{4}$.
- f(100)=$\frac{1}{100}$=$\frac{1}{100}$.
- f(0)=$\frac{1}{0}$=undefine (Problem, so we have to exclude it from the domain!)

- Domain $= \{x \mid x \neq 0\}$.
- Co–domain=$(-\infty, \infty)$.
- Range=$\{y \mid, y \neq 0\}$.

Finding Function Values

Recall

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

Example 4

Let $g(x) = x^2 - 2$. Find

- f(2)=$(2)^2 - 2$=2. (we replace each x with 2).
- f(u)=$(u)^2 - 2$=$u^2 - 2$.
- f(u^2)=$(u^2)^2 - 2$=$u^4 - 2$.
- f($u+1$)=$(u+1)^2 - 2$=$u^2 + 2u + 1 - 2 = u^2 + 2u - 1$.

Exercise 5

Let $f(x) = \frac{x-5}{x^2+3}$. Find

- $f(5)$.
- $f(2x)$.
- $f(x+h)$.
- $f(-7)$.

Example 6

Let $f(x) = x^2 + 2x$. Find $\frac{f(x+h)-f(x)}{h}$.

Solution:

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2 + 2(x+h) - (x^2+2x)}{h} \\
&= \frac{x^2 + 2xh + h^2 + 2x + 2h - x^2 - 2x}{h} \\
&= \frac{2xh + h^2 + 2h}{h} \\
&= \frac{h(2x+h+2)}{h} \\
&= 2x + h + 2
\end{aligned}$$

Exercise 7

Let $f(x) = 2x^2 - x + 1$. Find $\frac{f(x)-f(2)}{x-2}$.

2- The graph of a function

Example 8

Graph (sketch) the function $y = x^2 - 1$.

We substitute values of x to find the values of y and we fill the table

x	-2	-1	0	1	2	3
y						

Note:

- In ideal world, we will need to plot infinitely many points to get a perfect graph, but this is not possible, so our concern is only on the "general shape" of the function by joining only several points by a smooth curve whenever possible.
- In MATHS101, we will be able to graph more complicated functions in an easier way! (using calculus).

3 - Special functions

- $f(x) = c$ is called the constant function. The output is always the constant c and its graph is a horizontal line $y = c$.
- $f(x) = ax + b$ is called the linear function. The graph is always a straight line.
- $f(x) = ax^2 + bx + c$ is called the quadratic function. The graph is always a parabola.
- $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$ is called a polynomial in x.
- $f(x) = \frac{p(x)}{q(x)}$, where $p(x), q(x)$ are polynomials is called the rational function.
- $f(x) = \sqrt[n]{x}$, is called the root function.

Definition 9

A function f is called an algebraic function if it can be constructed using algebraic operations $(+, -, \cdot, \div, \sqrt[n]{})$ starting from polynomials.

Example: $g(x) = \frac{x^4 - x^2 + 1}{x + \sqrt[3]{x}} + (x+1)\sqrt{x+3}$ is an algebraic function.

Transcendental Functions

Definition 10

A function f is called an transcendental function if it is not algebraic.

These are transcendental functions:

- $f(x) = a^x$ is called the exponential function.
- $f(x) = \log_a x$ is called the logarithmic function.
- $f(x) = \sin x, \cos x, \tan x, \sec x, \cot x, \csc x$ are called the trigonometric function.
- $f(x) = \ln x$ is called the natural logarithmic function where $a = e = 2.71818182\ldots$.

Note: This course is early transcendental calculus course, meaning, we will study all those function right from the beginning.

Example 11

(Piecewise defined Functions)

$$g(x) = \begin{cases} x-1, & x \geq 3 \\ 3-x^2, & x < 3 \end{cases}$$

- g(1)=$3-(-1)^2$=2.
- g(-2)=$3-(-2)^2$=-1.
- g(6)=$6-1$=5.
- g(4)=$4-1$=3.
- g(3)=$3-1$=2.

Example 12

(Absolute Value Functions) Let $f(x) = |x|$ be the absolute value function. It can be written as piecewise function as follows:

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

- f(1)=1.
- f(-2)=2.
- f(-6)=6.
- f(0)=0.
- f(-3)=3.

Exercise 13

Sketch the graph of the absolute value function.

Factoring

1- Factoring by taking common factor:

- $3x + 6 = 3(x + 2)$.
- $x^2 + 6x = x(x + 6)$.
- $x^4 - 2x^3 + 8x^2 = x^2(x^2 - 2x + 8)$.
- $6x^4 + 12x^2 + 6x = 6x(x^3 + 2x + 1)$.
- $7x^5 - 7 = 7(x^5 - 1)$.

2- Factoring by grouping (works well if we have 4 terms):

- $3x^4 + 3x^3 + 7x + 7 = 3x^3(x + 1) + 7(x + 1) = (x + 1)(3x^3 + 7)$.
- $16x^3 - 28x^2 + 12x - 21 = 4x^2(4x - 7) + 3(4x - 7)$
  $= (4x - 7)(4x^2 + 3)$.
- $3xy + 2 - 3x - 2y = 3x(y - 1) + 2(1 - y) =$
  $3x(y - 1) - 2(y - 1) = (3x - 2)(y - 1)$.
- $4y^4 + y^2 + 20y^3 + 5y = y(4y^3 + y + 20y^2 + 5) =$
  $y(y(4y^2 + 1 + 5(4y^2 + 1)) = y(4y^2 + 1)(y + 5)$

Factoring Trinomial

Definition 14

A *trinomial* is an expression of the form $ax^2 + bx + c$.

To factor such a trinomial, we will use the quadratic formula of to get

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$

where α and β are the solution you will get from the quadratic formula.

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 15

Factor $8x^2 - 22x + 5$

Solution: Here we have $a = 8, b = -22, c = 5$, so we apply the quadratic formula to find α, β, so we have

$$\alpha, \beta = \frac{1}{4}, \frac{5}{2}$$

Hence

$$\begin{aligned} 8x^2 - 22x + 5 &= 8(x - \frac{1}{4})(x - \frac{5}{2}) \\ &= 8\frac{(4x-1)}{4}\frac{(2x-5)}{2} \\ &= (4x-1)(2x-5) \end{aligned}$$

Exercise 16

Factor each of the following trinomial expression completely:

1. $2x^2 + 13x - 7$
2. $3x^2 + 11x + 6$
3. $x^2 - 4$
4. $4x^2 - 25$
5. $-6x^2 - 13x + 5$
6. $x^2 + 12x + 36$

Solution:

1. $2x^2 + 13x - 7 = (2x - 1)(x + 7)$.
2. $3x^2 + 11x + 6 = (3x + 2)(x + 3)$.
3. $x^2 - 4 = (x - 2)(x + 2)$.
4. $4x^2 - 25 = (2x - 5)(2x + 5)$.
5. $-6x^2 - 13x + 5 = -(3x - 1)(2x + 5)$.
6. $x^2 + 12x + 36 = (x + 6)(x + 6)$

Factoring Cubes

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2$$

Example 17

- $x^3 - 8 = x^3 - 2^3 = (x-2)(x^2 + 2x + 4)$.
- $x^3 + 1 = x^3 + 1^3 = (x+1)(x^2 - x + 1)$.
- $64x^3 - 1 = 4^3x^3 - 1^3 = (4x-1)(16x^2 + 4x + 1)$.

Factoring higher degree

$$a^n - b^n = (a-b)(\underbrace{a^{n-1} + a^{n-2}b^2 + a^{n-3}b^2 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1}}_{n - \text{terms}}$$

Exercise 18

- $x^5 - 1 = x^5 - 1^5 = (x-1)(x^4 + x^3 + x^2 + x + 1)$.
- $x^7 + 1 = x^7 - (-1)^7 = (x-1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)$.
- $x^6 - 32 = x^6 - (2)^6 = (x-2)(x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32)$.